Initial Boundary Value Problems for Scalar and Vector Burgers Equations

By K. T. Joseph and P. L. Sachdev

In this article we study Burgers equation and vector Burgers equation with initial and boundary conditions. First we consider the Burgers equation in the quarter plane $x > 0$, $t > 0$ with Riemann type of initial and boundary conditions and use the Hopf–Cole transformation to linearize the problems and explicitly solve them. We study two limits, the small viscosity limit and the large time behavior of solutions. Next, we study the vector Burgers equation and solve the initial value problem for it when the initial data are gradient of a scalar function. We investigate the asymptotic behavior of this solution as time tends to infinity and generalize a result of Hopf to the vector case. Then we construct the exact N-wave solution as an asymptote of solution of an initial value problem extending the previous work of Sachdev et al. (1994). We also study the limit as viscosity parameter goes to 0. Finally, we get an explicit solution for a boundary value problem in a cylinder.

1. Introduction

The nonlinear parabolic partial differential equation

$$u_t + \frac{1}{2}(u^2)_x = \frac{\nu}{2}u_{xx} \tag{1}$$

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was first introduced by J. M. Burgers [1] as the simplest model for fluid flow. This equation describes the interplay between nonlinearity and diffusion, see Sachdev [2] and the references therein for physical interpretation and important solutions. A remarkable feature of this equation is that its solution with initial conditions of the form

$$u(x,0) = u_0(x) \tag{2}$$

can be explicitly written down. In fact Hopf [3] and Cole [4] independently showed that the equation (1) can be linearized through the transformation

$$u = -\nu \frac{v_x}{v}. \tag{3}$$

More precisely, assume that the initial data $u_0(x)$ are integrable in every finite interval and that

$$\int_0^x u_0(y)dy = o(|x|^2),$$

then Hopf [3] showed that if $v(x,t)$ satisfies the linear heat equation

$$v_t = \frac{\nu}{2} v_{xx} \tag{4}$$

with initial condition

$$v(x,0) = \exp\left[-\frac{1}{\nu}\int_0^y u_0(z)dz\right] \tag{5}$$

then $u(x,t)$ defined by (3) solves (1) and (2) and conversely, if $u(x,t)$ is a solution of (1) and (2), then $v(x,t)$ defined by (3) is a solution of (4) and (5) up to a time-dependent multiplicative factor that is irrelevent in (3). Solving for v from (4) and (5) and substituting it into (3), he obtained explicit formula for the solution of (1) and (2); namely,

$$u^\nu(x,t) = \frac{\int_R \left(\frac{x-y}{t}\right)\exp\left(-\frac{(x-y)^2}{2\nu t} - \frac{\int_0^y u_0(z)dz}{\nu}\right)dy}{\int_R \exp\left(-\frac{(x-y)^2}{2\nu t} - \frac{\int_0^y u_0(z)dz}{\nu}\right)dy} \tag{6}$$

and studied the asymptotic behavior of $u^\nu(x,t)$ as $t \to \infty$. He also constructed explicit weak entropy solution of the inviscid Burgers equation. Lighthill [5] discovered the N-wave solution of the Burgers equation (1); namely,

$$U^\infty(x,t) = \frac{x/t^{\frac{1}{2}}}{t^{\frac{1}{2}}\left[1 + \frac{t^{\frac{1}{2}}}{c_0}\exp\left(\frac{|x|^2}{2t\nu}\right)\right]}. \tag{7}$$

Sachdev et al. [6] showed that this solution can be obtained as time asymptotic of a pure initial value problem and found explicitly the constant c_0 in terms of one lobe area of the initial data.

In a more recent paper, [7] proposed the vector Burgers equation

$$U_t + U\nabla U = \frac{\nu}{2}\Delta U. \tag{8}$$

They observed that if we seek special solutions of the form

$$U = \nabla_x \phi \tag{9}$$

where $\phi(x, t)$ is a scalar function on $R^n \times [0, \infty)$, equation (8) leads to

$$\nabla\left[\phi_t + \frac{1}{2}|\nabla\phi|^2 - \frac{\nu}{2}\Delta\phi\right] = 0. \tag{10}$$

They used this observation to find special solutions of Burgers equation in cylindrical coordinates with axisymmetry.

Although there are many results for pure initial value problem, boundary value problem has been studied less. Explicit solutions of the Burgers equation (1) in the quarter plane with integrable initial data and piecewise constant boundary data were constructed by [8] using Hopf–Cole transformation. He obtained a formula for its weak limit as viscosity parameter goes to 0. Using maximum principle, this formula for weak limit was extended to general boundary data. When the initial and boundary data are two different constants, the problem was considered in some detail by [9] for ν small. Five distinct cases arose depending on the relative magnitudes of the constants appearing in the initial and boundary conditions. The solutions required the introduction of corner layers, shock layers, boundary layers, and transition layers. In the present article, we write an explicit form of the solution and recover various cases in the limit $\nu \to 0$. We also consider large time behavior of the solutions. Again, different cases arise depending on the relative magnitudes of the constants that appear in the initial and boundary conditions. It would be desirable to have solution for nonconstant initial boundary problem, but as [9] point out, that leads to an integral equation that seems difficult to solve. Using a generalized Hopf–Cole transformation, [10] and [11], in a series of papers, studied (1) with more general boundary conditions that include the flux condition at the origin. The asymptotic profile at infinity of the solution of (1) with flux condition was obtained by [12]. A table of solutions of the Burgers equation is contained in [13].

The aim of the present article is to study Burgers equation (1) and the vector Burgers equation (8) with initial and boundary conditions and investigate the asymptotic behavior as t tends to ∞ and as ν tends to 0. In Section 2, we treat the Burgers equation (1) in the quarter plane with Reimann type initial boundary data, solve it exactly, and study the asymptotic limits as t tends to ∞ or as the viscosity coefficient ν tends to 0. In Section 3 we study the vector Burgers equation, solve exactly the initial value problem for it when the initial data can be written as gradients of a scalar function and carry out the study of the limits as t tends to ∞ or ν tends to 0. We also study a boundary value problem for the vector Burgers equation in the same section.

2. Initial boundary value problem for Burgers equation and its asymptotics

In this section, we consider the Burgers equation

$$u_t + \frac{1}{2}(u^2)_x = \frac{\nu}{2}u_{xx} \tag{11}$$

in $x > 0$, $t > 0$, with initial condition

$$u(x,0) = u_I \tag{12}$$

and the boundary condition

$$u(0,t) = u_B, \tag{13}$$

where u_I and u_B are constants. Using standard Hopf–Cole transformation, we reduce this problem to a linear one and solve it explicitly. Before the statement of the result we introduce some notations:

$$\begin{aligned} A_-^\nu(x,t) &= \int_0^\infty \exp\Big(-\frac{(x-y)^2}{2t\nu} - \frac{u_I y}{\nu}\Big)dy, \\ A_+^\nu(x,t) &= \int_0^\infty \exp\Big(-\frac{(x+y)^2}{2t\nu} - \frac{u_I y}{\nu}\Big)dy, \\ B^\nu(x,t) &= \int_0^\infty \exp\Big(-\frac{(x+y)^2}{2t\nu} + \frac{u_B y}{\nu}\Big)dy. \end{aligned} \tag{14}$$

With these notations we prove the following result.

A solution of the initial boundary value problem (11)–(13) is given by

$$u^\nu(x,t) = u_B + \frac{(u_I - u_B)[A_-^\nu(x,t) - A_+^\nu(x,t)]}{A_-^\nu(x,t) + A_+^\nu(x,t) + 2u_B t\exp(-\frac{x^2}{2t\nu}) + (2u_B/\nu)(u_B t - x)B^\nu(x,t)}, \tag{15a}$$

when $u_I + u_B = 0$ and

$$u^\nu(x,t) = u_B + \frac{(u_I - u_B)[A_-^\nu(x,t) - A_+^\nu(x,t)]}{A_-^\nu(x,t) + \frac{(u_I-u_B)}{(u_I+u_B)}A_+^\nu(x,t) + \frac{2u_B}{(u_I+u_B)}B^\nu(x,t)}, \tag{15b}$$

when $u_I + u_B \neq 0$.

To prove this result, we use Hopf–Cole transformation. First, we set

$$w(x, t) = -\int_x^\infty [u(y, t) - u_I]dy. \tag{16}$$

From (11)–(13), we get

$$\begin{aligned} w_t + \frac{(w_x)^2}{2} + u_I w_x &= \frac{\nu}{2} w_{xx}, \\ w(x, 0) = 0, w_x(0, t) &= u_B - u_I. \end{aligned} \tag{17}$$

Now setting

$$v = \exp\left(-\frac{w}{\nu}\right), \tag{18}$$

we get from (17)

$$\begin{aligned} v_t + u_I v_x &= \frac{\nu}{2} v_{xx}, \\ \nu v_x(0, t) + (u_B - u_I)v(0, t) &= 0, \\ v(x, 0) &= 1. \end{aligned} \tag{19}$$

Making the final transformation

$$p = \exp\left(-\frac{u_I x}{\nu} + \frac{{u_I}^2 t}{2\nu}\right) v \tag{20}$$

we get from (19) the following problem for p:

$$\begin{aligned} p_t &= \frac{\nu}{2} p_{xx} \\ p(x, 0) &= \exp\left(-\frac{u_I x}{\nu}\right) \\ \nu p_x(0, t) + u_B p(0, t) &= 0. \end{aligned}$$

This problem has the explicit solution (see [8])

$$\begin{aligned} p(x, t) = {} & \frac{1}{(2\pi t\nu)^{1/2}} \int_0^\infty \left\{ \exp\left[-\frac{(x - y)^2}{2t\nu}\right] + \exp\left[-\frac{(x + y)^2}{2t\nu}\right]\right\} \\ & \cdot \exp\left(-\frac{u_I y}{\nu}\right) dy + \frac{2(u_B/\nu)}{(2\pi t\nu)^{1/2}} \\ & \cdot \int_0^\infty \int_y^\infty \exp\left[-\frac{u_B(y - z)}{\nu} - \frac{(x + z)^2}{2t\nu} - \frac{u_I y}{\nu}\right] dz dy. \end{aligned} \tag{21}$$

Now from (16), (18), and (20), we have

$$u^\nu(x,t) = w_x + u_I = -\nu(\log v)_x + u_I$$
$$= -\nu[(\log p)_x + u_I/\nu] + u_I = -\nu(\log p)_x.$$

In other words

$$u^\nu = -\nu \frac{p_x}{p}. \tag{22}$$

To write the formula in a convenient form we note that

$$\begin{aligned}
&\partial_x\left\{\int_y^\infty \exp\left[\frac{u_B z}{\nu} - \frac{(x+z)^2}{2t\nu}\right]dz\right\}\\
&= \int_y^\infty \exp\Big(\frac{u_B z}{\nu}\Big)\partial_z\left\{\exp\left[-\frac{(x+z)^2}{2t\nu}\right]\right\}dz\\
&= -\exp\left[\frac{u_B y}{\nu} - \frac{(x+y)^2}{2t\nu}\right] - \frac{u_B}{\nu}\left\{\int_y^\infty \exp\left[\frac{u_B z}{\nu} - \frac{(x+z)^2}{2t\nu}\right]dz\right\}. \qquad (23)
\end{aligned}$$

Here, the first equality is obtained by just writing the exponential of a sum as a product and using the fact that

$$\partial_x \exp\left[-\frac{(x+z)^2}{2\nu t}\right] = \partial_z \exp\left[-\frac{(x+z)^2}{2\nu t}\right],$$

and the second equality is obtained by the use of integration by parts. Similarly,

$$\begin{aligned}
&\partial_x\left\{\int_0^\infty \exp\left[\frac{-u_I y}{\nu} - \frac{(x+y)^2}{2t\nu}\right]dy\right\}\\
&\qquad = -\exp\Big(-\frac{x^2}{2t\nu}\Big) + \frac{u_I}{\nu}\left\{\int_0^\infty \exp\left[\frac{-u_I y}{\nu} - \frac{(x+y)^2}{2t\nu}\right]dy\right\}. \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_x\left\{\int_0^\infty \exp\left[\frac{-u_I y}{\nu} - \frac{(x-y)^2}{2t\nu}\right]dy\right\}\\
&\qquad = \exp\Big(-\frac{x^2}{2t\nu}\Big) - \frac{u_I}{\nu}\left\{\int_0^\infty \exp\left[\frac{-u_I y}{\nu} - \frac{(x-y)^2}{2t\nu}\right]dy\right\}. \qquad (25)
\end{aligned}$$

Now, it follows from (21), (23), (24), and (25) that

$$\begin{aligned}
p_x(x,t) = \frac{1}{-\nu(2\pi\nu t)^{\frac{1}{2}}}\Big[&u_I A_-^\nu(x,t) - u_I A_+^\nu(x,t)\\
&+ 2u_B A_+^\nu(x,t) + 2u_B \frac{u_B}{\nu} C^\nu(x,t)\Big]
\end{aligned}$$

and

$$p(x,t)=\frac{1}{(2\pi\nu t)^{\frac{1}{2}}}\left[A_-^\nu(x,t)+A_+^\nu(x,t)+\frac{2u_B}{\nu}C^\nu(x,t)\right],$$

where A_-^ν, A_+^ν and B^ν are given by (14), and

$$C^\nu(x,t)=\int_0^\infty\int_y^\infty \exp\left(-\frac{u_B(y-z)}{\nu}-\frac{(x+y)^2}{2t\nu}-\frac{u_I y}{\nu}\right)dzdy.$$

Substituting these formulas for p and p_x in (22), we get

$$\begin{aligned}u^\nu(x,t)&=\frac{u_I A_-^\nu(x,t)-u_I A_+^\nu(x,t)+2u_B A_+^\nu(x,t)+u_B.\frac{2u_B}{\nu}C^\nu(x,t)}{A_-^\nu(x,t)+A_+^\nu(x,t)+\frac{2u_B}{\nu}C^\nu(x,t)}\\&=\frac{(u_I-u_B)\left[A_-^\nu(x,t)-A_+^\nu(x,t)\right]+u_B\left[A_-^\nu(x,t)+A_+^\nu(x,t)+\frac{2u_B}{\nu}C^\nu(x,t)\right]}{A_-^\nu(x,t)+A_+^\nu(x,t)+\frac{2u_B}{\nu}C^\nu(x,t)}\\&=\frac{(u_I-u_B)\left[A_-^\nu(x,t)-A_+^\nu(x,t)\right]}{A_-^\nu(x,t)+A_+^\nu(x,t)+\frac{2u_B}{\nu}C^\nu(x,t)}+u_B.\end{aligned}\tag{26}$$

Now we express C^ν in terms of A_+^ν and B^ν. There are two cases to consider. The first one is when $u_I+u_B=0$. In this case,

$$\begin{aligned}C^\nu(x,t)&=\int_0^\infty\int_y^\infty \exp\left[-\frac{u_B(y-z)}{\nu}-\frac{u_I y}{\nu}-\frac{(x+z)^2}{2t\nu}\right]dzdy\\&=\int_0^\infty 1.\int_y^\infty \exp\left[\frac{u_B z}{\nu}-\frac{(x+z)^2}{2t\nu}\right]dzdy\\&=\int_0^\infty y\exp\left[\frac{u_B y}{\nu}-\frac{(x+y)^2}{2t\nu}\right]dy\\&=-t\nu\int_0^\infty \partial_y\exp\left[\frac{u_B y}{\nu}-\frac{(x+y)^2}{2t\nu}\right]dy\\&\qquad+(u_B t-x)\int_0^\infty \exp\left[\frac{u_B y}{\nu}-\frac{(x+y)^2}{2t\nu}\right]dy\\&=t\nu\exp\left(\frac{-x^2}{2t\nu}\right)+(u_B t-x)\int_0^\infty \exp\left[\frac{u_B y}{\nu}-\frac{(x+y)^2}{2t\nu}\right]dy\\&=t\nu\exp\left(-\frac{x^2}{2t\nu}\right)+(u_B t-x)B^\nu(x,t).\end{aligned}\tag{27}$$

When $u_I + u_B \neq 0$,

$$
\begin{aligned}
C^\nu(x,t) &= \int_0^\infty \int_y^\infty \exp\Big[-\frac{u_B(y-z)}{\nu} - \frac{u_I y}{\nu} - \frac{(x+z)^2}{2t\nu}\Big] dz dy \\
&= \int_0^\infty \exp\Big[-\frac{(u_I+u_B)y}{\nu}\Big] \int_y^\infty \exp\Big[\frac{u_B z}{\nu} - \frac{(x+z)^2}{2t\nu}\Big] dz dy \\
&= -\frac{\nu}{(u_I+u_B)} \int_0^\infty \partial_y \exp\Big[-\frac{(u_I+u_B)y}{\nu}\Big] \\
&\qquad \cdot \int_y^\infty \exp\Big[\frac{u_B z}{\nu} - \frac{(x+z)^2}{2t\nu}\Big] dz dy \\
&= \frac{\nu}{(u_I+u_B)} \Big\{ \int_0^\infty \exp\Big[\frac{u_B y}{\nu} - \frac{(x+y)^2}{2t\nu}\Big] dy \\
&\qquad - \int_0^\infty \exp\Big[-\frac{u_I y}{\nu} - \frac{(x+y)^2}{2t\nu}\Big] dy \Big\} \\
&= \frac{\nu}{(u_I+u_B)} [B^\nu(x,t) - A_+^\nu(x,t)].
\end{aligned}
\tag{28}
$$

Using (27) for the case $u_I + u_B = 0$ and (28) for the case $u_I + u_B \neq 0$ in (26), the formula (25) for u^ν follows.

2.1. *Study of the limit as $\nu \to 0$*

Here, we compute the pointwise limit of $u^\nu(x,t)$ as the viscosity parameter $\nu \to 0$. Let $H(x)$ be the usual Heaviside function and $s = (u_I + u_B)/2$, then for $x \geq 0$, $t \geq 0$, we have the following formula for the pointwise limit function $u(x,t) = \lim_{\nu \to 0} u^\nu(x,t)$.

Case 1. $u_B > u_I$ and $(u_I + u_B) > 0$

$$u(x,t) = u_I H(x - st) + u_B H(st - x)$$

Case 2. $u_I > 0$, $u_B > 0$ and $u_I > u_B$

$$u(x,t) = u_B H(u_B t - x) + (x/t) H(u_I t - x) H(x - u_B t) + u_I H(x - u_I t)$$

Case 3. $u_I < 0$ and $(u_I + u_B) \leq 0$

$$u(x,t) = u_I$$

Case 4. $u_I = 0$ and $u_B < 0$ or $u_I > 0$ and $u_B \leq 0$ and $u_I + u_B \neq 0$ or $u_I > 0$ and $u_I + u_B = 0$,

$$u(x,t) = (x/t) H(u_I t - x) + u_I H(x - u_I t).$$

To study the $\lim_{\nu\to 0} u^\nu(x,t)$, we write (15) for u^ν in terms of the standard "*erfc*" and use its asymptotic form. For this, we denote

$$erfc(y) = \int_y^\infty \exp(-y^2)dy. \tag{29}$$

$$\begin{aligned} A_-^\nu(x,t) &= \exp\left(\frac{u_I^2 t}{2\nu} - \frac{u_I x}{\nu}\right)\int_0^\infty \exp\left[-\frac{(y-x+u_I t)^2}{2t\nu}\right]dy \\ &= (t\nu)^{(1/2)}\exp\left(\frac{u_I^2 t}{2\nu} - \frac{u_I x}{\nu}\right)\int_{\frac{-x+u_I t}{(2t\nu)^{1/2}}}^\infty \exp(-y^2)dy \\ &= (t\nu)^{(1/2)}\exp\left(\frac{u_I^2 t}{2\nu} - \frac{u_I x}{\nu}\right) erfc\left(\frac{-x+u_I t}{(2\nu t)^{1/2}}\right). \end{aligned} \tag{30}$$

Similarly,

$$A_+^\nu(x,t) = (t\nu)^{(1/2)}\exp\left(\frac{u_I^2 t}{2\nu} + \frac{u_I x}{\nu}\right) erfc\left[\frac{x+u_I t}{(2\nu t)^{1/2}}\right], \tag{31}$$

and

$$B^\nu(x,t) = (t\nu)^{(1/2)}\exp\left(\frac{u_B^2 t}{2\nu} - \frac{u_B x}{\nu}\right) erfc\left[\frac{x-u_B t}{(2\nu t)^{1/2}}\right]. \tag{32}$$

To study the asymptotic behavior of u^ν as $\nu \to \infty$ we study the behavior of A_-^ν, A_+^ν and B^ν using the asymptotic expansions of the *erfc*; namely,

$$erfc(y) = \left[\frac{1}{2y} - \frac{1}{4y^3} + o\left(\frac{1}{y^3}\right)\right]\exp(-y^2),\ y \to \infty \tag{33}$$

and

$$erfc(-y) = (\pi)^{(1/2)} - \left[\frac{1}{2y} - \frac{1}{4y^3} + o\left(\frac{1}{y^3}\right)\right]\exp(-y^2),\ y \to \infty. \tag{34}$$

From (30)–(34), we have the following as $\nu \to 0$. If $-x + u_I t > 0$

$$A_-^\nu(x,t) \approx \frac{(t\nu)}{-x+u_I t}\exp\left(-\frac{x^2}{2\nu t}\right) \tag{35a}$$

and for $-x + u_I t < 0$

$$A_-^\nu(x,t) \approx (2\pi t\nu)^{(1/2)}\exp\left(\frac{u_I^2 t}{2\nu} - \frac{u_I x}{\nu}\right) - \frac{(t\nu)}{-x+u_I t}\exp\left(-\frac{x^2}{2\nu t}\right) \tag{35b}$$

For $x + u_I t > 0$

$$A_+^\nu(x,t) \approx \frac{(t\nu)}{x+u_I t}\exp\left(-\frac{x^2}{2\nu t}\right) \tag{35c}$$

and for $x + u_I t < 0$

$$A_+^\nu(x,t) \approx (2\pi t\nu)^{(1/2)} \exp\left(\frac{u_I^2 t}{2\nu} + \frac{u_I x}{\nu}\right) - \frac{(t\nu)}{x+u_I t} \exp\left(-\frac{x^2}{2\nu t}\right) \tag{35d}$$

For $x - u_B t > 0$

$$B^\nu(x,t) \approx \frac{(t\nu)}{x - u_B t} \exp\left(-\frac{x^2}{2\nu t}\right) \tag{35e}$$

and for $x - u_B t < 0$

$$B^\nu(x,t) \approx (2\pi t\nu)^{(1/2)} \exp\left(\frac{u_B^2 t}{2\nu} - \frac{u_B x}{\nu}\right) - \frac{(t\nu)}{x-u_B t} \exp\left(-\frac{x^2}{2\nu t}\right) \tag{35f}$$

There are several cases to consider. First, we consider

Case 1. $u_B > u_I$, $u_I + u_B > 0$.
First we take up the subcase $u_B > 0$, $u_I < 0$, and $u_I + u_B > 0$. The line $x = (u_I + u_B)/2t$ divides the quarter plane $x > 0$, $t > 0$ into two regions. In the region $x < (u_I + u_B)/2t$, $x - u_B t \leq (u_I - u_B)/2t < 0$. Using the asymptotic formulae (35b)–(35e) in (15b) we get, as $\nu \to 0$,

$$u^\nu(x,t) \approx u_B + \frac{(u_I - u_B)[\pi^{1/2} - O(1)\exp(\frac{u_I x}{\nu})]}{\pi^{1/2} + O(1)\exp(\frac{u_I x}{\nu}) + \frac{2u_B}{(u_I+u_B)}\pi^{1/2}\exp\frac{(u_B-u_I)}{2\nu}[-x + \frac{(u_I+u_B)}{2}t]}.$$

Because $u_I < 0$ and $u_B - u_I > 0$, we get, for $x < (u_I + u_B)/2t$,

$$\lim_{\nu\to 0} u^\nu(x,t) = u_B.$$

By a similar argument, we have, for $x > (u_I + u_B)/2t$ and $x < u_B t$,

$$u^\nu(x,t) \approx u_B + \frac{(u_I - u_B)[\pi^{1/2} - O(1)\exp(\frac{u_I x}{\nu})]}{\pi^{1/2} + O(1)\exp(\frac{u_I x}{\nu}) + \frac{2u_B}{(u_I+u_B)}\pi^{1/2}\exp\frac{(u_B-u_I)}{2\nu}[-x + \frac{(u_I+u_B)}{2}t]},$$

so that

$$\lim_{\nu\to 0} u^\nu(x,t) = u_B + (u_I - u_B) = u_I.$$

If $x > (u_I + u_B)t/2$ and $x > u_B t$, as before, from (15b) and (35b)–(35e), we get

$$u^\nu(x,t) \approx u_B + \frac{(u_I - u_B)[\pi^{1/2} - O(1)\exp(\frac{u_I x}{\nu})]}{\pi^{1/2} + O(1)\exp(\frac{u_I x}{\nu}) + O(\nu^{1/2})},$$

so that

$$\lim_{\nu\to 0} u^\nu(x,t) = u_B + (u_I - u_B) = u_I.$$

The other subcases $u_B > 0$, $u_I > 0$, $u_I < u_B$, and $u_B > 0$, $u_I = 0$ are similar.

Case 2. $u_I > 0, u_B > 0, u_B < u_I$.
In the region $x - u_B t < 0$, we have $-x + u_I t > 0$, and so from (15b) and (35a), (35c), and (35f) we get

$$u^\nu(x,t) \approx u_B + \frac{(u_I - u_B)\Big[\frac{(2t\nu)^{1/2}}{2(-x+u_I t)} - \frac{(2t\nu)^{1/2}}{2(x+u_I t)}\Big]}{\frac{(2t\nu)^{1/2}}{2(-x+u_I t)} + \frac{u_I - u_B}{u_I + u_B}\cdot\frac{(2t\nu)^{1/2}}{2(x+u_I t)} + \frac{2u_B \pi^{1/2}}{u_I + u_B}\exp\frac{(x-u_B t)^2}{2t\nu}}.$$

Therefore, for $x < u_B t$, we have

$$\lim_{\nu\to 0} u^\nu(x,t) = u_B.$$

On the other hand, in the region $x > u_I t$, we have $x > u_B t$ so that

$$u^\nu(x,t) \approx u_B + \frac{(u_I - u_B)\Big[\pi^{1/2} - \frac{(2t\nu)^{1/2}}{2(x+u_I t)}\exp\Big(-\frac{(x-u_I t)^2}{2t\nu}\Big)\Big]}{\pi^{1/2} - \frac{(2t\nu)^{1/2}}{2(x+u_I t)}\frac{(u_I - u_B)}{(u_I + u_B)}\exp\Big[-\frac{(x-u_I t)^2}{2t\nu}\Big] + \frac{(2t\nu)^{1/2}}{(x-u_B t)}\frac{2u_B}{u_I+u_B}\exp\Big(-\frac{(x-u_I t)^2}{2t\nu}\Big)}$$

and we get

$$\lim_{\nu\to 0} u^\nu(x,t) = u_I.$$

In the region $u_B t < x < u_I t$, we have

$$u^\nu(x,t) \approx u_B + \frac{(u_I - u_B)\Big[\frac{(2t\nu)^{1/2}}{2(-x+u_I t)}\exp(-\frac{x^2}{2t\nu}) - \frac{(2t\nu)^{1/2}}{2(x+u_I t)}\exp(-\frac{x^2}{2t\nu})\Big]}{\frac{(2t\nu)^{1/2}}{2(-x+u_I t)}\exp(-\frac{x^2}{2t\nu}) + \frac{(u_I-u_B)}{(u_I+u_B)}\frac{(2t\nu)^{1/2}}{2(x+u_I t)}\exp\big(-\frac{x^2}{2t\nu}\big) + \frac{(2t\nu)^{1/2}}{2(x-u_B t)}\frac{2u_B}{u_I+u_B}\exp -\big(\frac{x^2}{2t\nu}\big)}$$

and we find that

$$\lim_{\nu\to 0} u^\nu(x,t) = u_B + \frac{(u_I - u_B)\big[\frac{1}{-x+u_I t} - \frac{1}{x+u_I t}\big]}{\frac{1}{-x+u_I t} + \frac{u_I-u_B}{u_I+u_B}\cdot\frac{1}{x+u_I t} + \frac{2u_B}{u_I+u_B}\cdot\frac{1}{x-u_B t}}.$$

On simplification, we have, for $u_B t < x < u_I t$,

$$\lim_{\nu\to 0} u^\nu(x,t) = \frac{x}{t}.$$

This completes case 2.

Case 3. $u_I < 0$, $u_I + u_B \leq 0$.

First we consider the subcase $u_I < 0$, $u_B < 0$, $u_I - u_B < 0$. We get from (15b) and (35b)–(35e),

$$u^\nu(x, t) = u_B + \frac{(u_I - u_B)\left[\pi^{1/2} - O(1)\exp(\frac{2u_I x}{\nu})\right]}{\pi^{1/2} + O(1)\frac{u_I - u_B}{u_I + u_B}\exp(\frac{2u_I x}{\nu}) + O(1)\frac{2u_B}{u_I + u_B}\exp\left(-\frac{(x-u_I t)^2}{2\nu t}\right)},$$

so that, for $x > 0$,

$$\lim_{\nu \to 0} u^\nu(x, t) = u_B + (u_I - u_B) = u_I.$$

For the other subcases the analysis is similar and so we omit the details.

Case 4. $u_I = 0$ and $u_B < 0$ or $u_I > 0$ and $u_B \leq 0$ and $u_I + u_B$ non-zero or $u_I > 0$ and $u_I + u_B = 0$.

For the first subcase, for $x > 0$, we get

$$u^\nu(x, t) \approx u_B - u_B \frac{\left[\pi^{1/2} + \frac{(2t\nu)^{1/2}}{2x}\exp(-\frac{x^2}{2t\nu})\right]}{\pi^{1/2} - \frac{(2t\nu)^{1/2}}{2x}\exp\left(-\frac{x^2}{2t\nu}\right) + \frac{(2t\nu)^{1/2}}{2(x-u_B t)}\exp\left(-\frac{x^2}{2t\nu}\right)},$$

so that

$$\lim_{\nu \to 0} u^\nu(x, t) = u_B - u_B = 0.$$

For the subcase $u_I > 0$, $u_B < 0$, $u_I + u_B$ not zero, following the previous analysis, we have in the region $x > u_I t$,

$$u^\nu(x, t) = u_B + \frac{(u_I - u_B)\left[\pi^{1/2} - \frac{(2t\nu)^{1/2}}{2(x+u_I t)}\exp\left(-\frac{(x-u_I t)^2}{2t\nu}\right)\right]}{\pi^{1/2} + \frac{u_I - u_B}{u_I + u_B}\frac{(2t\nu)^{1/2}}{2(x+u_I t)}\exp\left[-\frac{(x-u_I t)^2}{2t\nu}\right] + \frac{2u_B}{u_I + u_B}\frac{(2t\nu)^{1/2}}{2(x+u_B t)}\exp\left[-\frac{(x-u_I t)^2}{2t\nu}\right]}.$$

So we get for $x > u_I t$,

$$\lim_{\nu \to 0} u^\nu(x, t) = u_I.$$

In the region $x < u_I t$, we have

$$\lim_{\nu \to 0} u^\nu(x, t) = u_B + \frac{(u_I - u_B)\left[\frac{1}{-x+u_I t} - \frac{1}{x+u_I t}\right]}{\frac{1}{-x+u_I t} + \frac{u_I - u_B}{u_I + u_B}\frac{1}{x+u_I t} + \frac{2u_B}{u_I + u_B}\frac{1}{x-u_B t}} = \frac{x}{t}.$$

The other subcases can be treated similarly. This completes the proof of the result.

2.2. Large time behavior

Next, we study the limit of $u^{\nu}(x,t)$ as $t \to \infty$ where $u(x,t)$ is the solution of (1)–(3) given by $(15)_j$, $j = 1, 2$ with a fixed coefficient of viscosity $\nu > 0$. We prove the following result.

1. If $u_I < 0$, $u_B < 0$ and $u_I > u_B$, then

$$\lim_{t\to\infty} u(x,t) = -u_I \coth\left(\frac{u_I x}{\nu} + c\right), c = \frac{1}{2}\log\left(\frac{u_B - u_I}{u_I + u_B}\right). \tag{36}$$

2. If $u_I < 0$, $u_I < u_B$ and $u_B + u_I < 0$, then

$$\lim_{t\to\infty} u(x,t) = -u_I \tanh\left(\frac{u_I x}{\nu} + c\right), c = \frac{1}{2}\log\left(\frac{u_I - u_B}{u_I + u_B}\right). \tag{37}$$

3. If $u_I = 0$ and $u_B < 0$, we have

$$\lim_{t\to\infty} u(x,t) = \frac{u_B}{1 - u_B \frac{x}{\nu}}. \tag{38}$$

4. For all other cases,

$$\lim_{t\to\infty} u(x,t) = u_B. \tag{39}$$

To prove the result first, we get the asymptotic behavior of A^{ν}_{-}, A^{ν}_{+}, and B^{ν} as t tends to infinity. Using the asymptotic expansions (33) and (34) in (30)–(32) we have, as $t \to \infty$,

$$A^{\nu}_{-}(x,t) \approx \frac{(t\nu)}{-x + u_I t} \exp\left(-\frac{x^2}{2\nu t}\right), u_I > 0 \tag{40a}$$

$$A^{\nu}_{-}(x,t) \approx (2\pi t\nu)^{(1/2)} \exp\left(\frac{u_I^2 t}{2\nu} - \frac{u_I x}{\nu}\right) - \frac{(t\nu)}{-x + u_I t} \exp\left(-\frac{x^2}{2\nu t}\right), u_I < 0 \tag{40b}$$

$$A^{\nu}_{+}(x,t) \approx \frac{(t\nu)}{x + u_I t} \exp\left(-\frac{x^2}{2\nu t}\right), u_I > 0 \tag{40c}$$

$$A^{\nu}_{+}(x,t) \approx (2\pi t\nu)^{(1/2)} \exp\left(\frac{u_I^2 t}{2\nu} + \frac{u_I x}{\nu}\right) - \frac{(t\nu)}{x + u_I t} \exp\left(-\frac{x^2}{2\nu t}\right), u_I < 0 \tag{40d}$$

$$B^{\nu}(x,t) \approx \frac{(t\nu)}{x - u_B t} \exp\left(-\frac{x^2}{2\nu t}\right), u_B < 0 \tag{40e}$$

and

$$B^\nu(x,t) \approx (2\pi t\nu)^{(1/2)} \exp\left(\frac{u_B^2 t}{2\nu} - \frac{u_B x}{\nu}\right) - \frac{(t\nu)}{x - u_B t} \exp\left(-\frac{x^2}{2\nu t}\right), u_B > 0. \tag{40f}$$

$$A_-^\nu(x,t) \approx \frac{\pi^{1/2}}{2} - \frac{x}{(2t\nu)^{1/2}}, u_I = 0, \tag{40g}$$

$$A_-^\nu(x,t) \approx \frac{\pi^{1/2}}{2} + \frac{x}{(2t\nu)^{1/2}}, u_I = 0, \tag{40h}$$

$$B^\nu(x,t) \approx \frac{\pi^{1/2}}{2} + \frac{x}{(2t\nu)^{1/2}}, u_B = 0. \tag{40i}$$

First, we consider the case $u_I < 0$, $u_B < 0$. Using (40b), (40d), and (40e) in (15b), we get

$$u(x,t) \approx u_B + \frac{(u_I - u_B)\left[\pi^{1/2}\exp\left(\frac{u_I^2 t}{2\nu} - \frac{u_I x}{\nu}\right) - \pi^{1/2}\exp\left(\frac{u_I^2 t}{2\nu} + \frac{u_I x}{\nu}\right)\right]}{\pi^{1/2}\exp\left(\frac{u_I^2 t}{2\nu} - \frac{u_I x}{\nu}\right) + \pi^{1/2}\frac{u_I - u_B}{u_I + u_B}\exp\left(\frac{u_I^2 t}{2\nu} + \frac{u_I x}{\nu}\right) + \frac{2u_B}{u_I + u_B}\frac{(2\nu t)^{1/2}}{x - u_B t}\exp\left(-\frac{x^2}{2\nu t}\right)}.$$

So, in this case, we get

$$\lim_{t\to\infty} u(x,t) = u_B + \frac{(u_I - u_B)[\exp(-\frac{u_I x}{\nu}) - \exp(\frac{u_I x}{\nu})]}{\exp(-\frac{u_I x}{\nu}) + \frac{u_I - u_B}{u_I + u_B}\exp(\frac{u_I x}{\nu})}.$$

On simplification we get

$$\lim_{t\to\infty} u(x,t) = u_I \frac{\exp(-\frac{u_I x}{\nu}) - \frac{u_I - u_B}{u_I + u_B}\exp(\frac{u_I x}{\nu})}{\exp(-\frac{u_I x}{\nu}) + \frac{u_I - u_B}{u_I + u_B}\exp(\frac{u_I x}{\nu})}. \tag{41}$$

Now, if $u_I > u_B$, by rewriting (41) we get (36), the first part of the result. If $u_I < u_B$, we get (37) of the part two.

Now consider $u_I < 0$ and $u_B = 0$. Using (40b), (40d), and (40i) in (15b), we get

$$\lim_{t\to\infty} u(x,t) = u_I \frac{\exp(-\frac{u_I x}{\nu}) - \exp(\frac{u_I x}{\nu})}{\exp(-\frac{u_I x}{\nu}) + \exp(\frac{u_I x}{\nu})} = -u_I \tanh\left(\frac{u_I x}{\nu}\right).$$

Now, consider the case $u_I < 0$, $u_B > 0$ and $u_I + u_B \neq 0$. As before, we have from (15b) and (40b), (40d), and (40f) the limit

$$\lim_{t\to\infty} u(x,t) = u_B + \frac{(u_I - u_B)[\exp(-\frac{u_I x}{\nu}) - \exp(\frac{u_I x}{\nu})]}{\exp(-\frac{u_I x}{\nu}) + \frac{u_I - u_B}{u_I + u_B}\exp(\frac{u_I x}{\nu}) + \frac{2u_B}{u_I + u_B}\exp(-\frac{u_B x}{\nu})\exp(\frac{(u_B^2 - u_I^2)t}{2\nu})}. \tag{42}$$

Therefore,

$$\lim_{t\to\infty} u(x,t) = u_I \frac{\exp(-\frac{u_I x}{\nu}) - \frac{u_I - u_B}{u_I + u_B}\exp(\frac{u_I x}{\nu})}{\exp(-\frac{u_I x}{\nu} + \frac{u_I - u_B}{u_I + u_B}\exp(\frac{u_I x}{\nu})}$$

provided $u_B + u_I < 0$, because in this case, $(u_B^2 - u_I^2) < 0$. On simplification, we get (37). When $u_I + u_B > 0$, we have $u_B^2 - u_I^2 > 0$, and so (42) gives

$$\lim_{t\to\infty} u^\nu(x,t) = u_B.$$

Now, take up the case $u_I < 0$, $u_B > 0$ and $u_I + u_B = 0$. From (15a) and (40b), (40d), and (40f) (after dividing numerator and dinominator of the resulting expression by $\pi^{1/2}(2\pi t\nu)^{1/2}\exp(\frac{u_I^2 t}{\nu})$ we get

$$u^\nu(x,t) \approx u_B + \frac{(u_I - u_B)[\exp(-\frac{u_I x}{\nu}) - \exp(\frac{u_I x}{\nu})]}{\exp(-\frac{u_I x}{\nu}) + \exp(\frac{u_I x}{\nu}) + \frac{2u_B t}{\pi^{1/2}(2\pi t\nu)^{1/2}}\exp(-\frac{x^2}{2\nu t} - \frac{u_I^2 t}{2\nu}) + \frac{2u_B(u_B t - x)}{\nu}\exp(-\frac{u_B x}{\nu})},$$

so that

$$\lim_{t\to\infty} u^\nu(x,t) = u_B.$$

When $u_I = 0$ and $u_B < 0$, using (40f), (40g), and (40h) in (15b), we get

$$u^\nu(x,t) \approx u_B - \frac{u_B\Big[2.(2\pi t\nu)^{1/2}\frac{x}{(2\nu t)^{1/2}}\Big]}{2(2\pi t\nu)^{1/2}\frac{x}{(2\nu t)^{1/2}} + \frac{2\pi^{1/2}(t\nu)}{x - u_B t}}. \tag{43}$$

From (43), we get for $u_I = 0$ and $u_B < 0$,

$$\lim_{t\to\infty} u^\nu(x,t) = u_B\left[1 - \frac{x}{x - \frac{\nu}{u_B}}\right] = \frac{u_B}{1 - \frac{u_B x}{\nu}}.$$

Now, we consider the case $u_I > 0$. First let $u_B < 0$ and $u_I + u_B \neq 0$. We have from (15b) and (40a), (40c), and (40e) after cancelling out common terms,

$$\lim_{t\to\infty} u^\nu(x,t) = u_B + \frac{(u_I - u_B)\Big[\frac{\nu}{u_I - x/t} - \frac{\nu}{u_I + x/t}\Big]}{(u_I - u_B)\Big[\frac{\nu}{u_I - x/t} + \frac{u_I - u_B}{u_I + u_B}\frac{\nu}{u_I + x/t} + \frac{2u_B}{u_I + u_B}\frac{t\nu}{x - u_B t}\Big]} = u_B.$$

The case $u_I + u_B = 0$ is similar; here we use (15a) instead of (15b). Now, consider the case $u_I > 0$, $u_B > 0$. As before, using the asymptotics

(40a), (40c), and (40f) in (15b) and dividing both numerator and dinominator by $\pi^{1/2}(t\nu)\exp(-x^2/2\nu t)$ in the resulting expression, we get

$$\lim_{t\to\infty} u^\nu(x,t)$$

$$= u_B + \frac{(u_I - u_B)\left[\frac{1}{u_I t - x} - \frac{1}{u_I t + x}\right]}{(u_I - u_B)\left[\frac{1}{u_I t - x} + \frac{u_I - u_B}{u_I + u_B}\frac{1}{u_I t + x} + \frac{2u_B}{u_I + u_B}(2\pi)^{1/2}\exp(\frac{(x-u_B t)^2}{2\nu t})\right]} = u_B.$$

The case $u_I > 0$ and $u_B = 0$ is similar. This completes the proof of the result.

Consider the stationary problem for $0 < x < \infty$

$$\nu q_{xx} = (q^2)_x,\ q(0) = u_B,\ q(\infty) = u_I.$$

It can be easily checked that this problem has solution if $u_I < 0$ and $u_B < u_I$ or $u_I < 0$ and $u_I + u_B < 0$ or $u_I = 0$ and $u_B < 0$. Solving the problem explicitly, we get the functions on the right-hand side of (36)–(38). Here, we have shown that they are time asymptotes of boundary value problems (11)–(13).

3. Higher dimensional extensions

Consider vector equivalent of Burgers equation studied by [7]; namely,

$$U_t + U.\nabla U = \frac{\nu}{2}\Delta U. \tag{44}$$

[7] observed that if we seek a solution U of (44), which is gradient of some scalar function ϕ,

$$U = \nabla_x \phi, \tag{45}$$

then equation (44) becomes

$$\nabla_x\left[\phi_t + \frac{|\nabla\phi|^2}{2} - \frac{\nu}{2}\Delta\phi\right] = 0.$$

This leads to

$$\phi_t + \frac{|\nabla\phi|^2}{2} - \frac{\nu}{2}\Delta\phi = f(t). \tag{46}$$

where $f(t)$ is an arbitrary function of t. Because we are interested in the space derivative $\nabla_x\phi$, and this is independent of $f(t)$, we let $f(t) = 0$. If we are given initial data for U that are gradients of some scalar function ϕ_0 of the form,

$$U(x,0) = \nabla_x \phi_0(x) \tag{47}$$

it is enough to find a solution ϕ of

$$\phi_t + \frac{|\nabla\phi|^2}{2} - \frac{\nu}{2}\Delta\phi = 0 \tag{48}$$

with initial condition

$$\phi(x, 0) = \phi_0(x). \tag{49}$$

We may then use (45) to get the solution U of (44) and (47). To solve (48) and (49) we use the Hopf–Cole transformation

$$\theta(x, t) = \exp\left(-\frac{\phi}{\nu}\right). \tag{50}$$

Using (50) in (48) and (49), we get the linear problem

$$\theta_t = \frac{\nu}{2}\Delta\theta, \tag{51}$$

$$\theta(x, 0) = \exp\left(-\frac{\phi_0(x)}{\nu}\right). \tag{52}$$

Solving (51) and (52), we have

$$\theta^\nu(x, t) = \frac{1}{(2\pi\nu t)^{\frac{n}{2}}}\int_{R^n} \exp\left(-\frac{|x-y|^2}{2\nu t} - \frac{\phi_0(y)}{\nu}\right)dy. \tag{53}$$

From (45), (50), and (52), we see that the solution to (44) and (47) is given by

$$U^\nu = -\nu\frac{\nabla\theta^\nu}{\theta^\nu}. \tag{54}$$

From (53) and (54), it follows that

$$U^\nu(x, t) = \frac{\int_{R^n}\left(\frac{x-y}{t}\right)\exp\left(-\frac{|x-y|^2}{2\nu t} - \frac{\phi_0(y)}{\nu}\right)dy}{\int_{R^n}\exp\left(-\frac{|x-y|^2}{2\nu t} - \frac{\phi_0(y)}{\nu}\right)dy}. \tag{55}$$

First, we study the asymptotic behavior of this solution as $t \to \infty$.

3.1. Asymptotic behavior as $t \to \infty$, a general result

Here, we assume initial data of the form (47)

$$U^\nu(x,0) = \nabla_x \phi_0(x),$$

where ϕ_0 satisfies the estimate

$$\phi_0(x) = \sum_1^n \phi_0^i(x_i) + o(1), |x| \to \infty, \tag{56}$$

where $\phi_0^i(x), i = 1,2,\ldots,n$ are differentiable functions from R^1 to R^1. Assume further that both the limits

$$\lim_{x_i \to -\infty} \phi_0^i(x_i) = \phi_i^-, \lim_{x_i \to \infty} \phi_0^i(x_i) = \phi_i^+ \tag{57}$$

exist. With the notations

$$\xi = \frac{x}{(\nu t)^{\frac{1}{2}}}, k_i = \frac{(\phi_i^+ - \phi_i^-)}{\nu} \tag{58}$$

and

$$g_i(\xi_i) = \exp\left(-\frac{k_i}{2}\right)\int_{-\infty}^{\xi_i} \exp\left(-\frac{z_i^2}{2}\right)dz_i + \exp\left(\frac{k_i}{2}\right)\int_{\xi_i}^{\infty} \exp\left(-\frac{z_i^2}{2}\right)dz_i, \tag{59}$$

for $i = 1,2,\ldots,n$, we prove the following result.

Let $U^\nu(x,t)$ be the solution of (44) and (47) given by the formula (53) and (54) with ϕ_0 satisfying the conditions (56) and (57), then we have the following limit as $t \to \infty$ uniformly in ξ in bounded subsets of R^n:

$$\lim_{t\to\infty}\left(\frac{t}{\nu}\right)^{\frac{1}{2}} U\left((\nu t)^{\frac{1}{2}}\xi, t\right) = -\left(\frac{g_1'(\xi_1)}{g_1(\xi_1)}, \frac{g_2'(\xi_2)}{g_2(\xi_2)}, \ldots, \frac{g_n'(\xi_n)}{g_n(\xi_n)}\right). \tag{60}$$

To prove this result, we follow [3]. Consider θ^ν given by (53), where ϕ_0 satisfies the conditions (56) and (57). After a change of variable, the expression for θ^ν becomes

$$\theta^\nu(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n} \exp\left(-\frac{|z|^2}{2} - \frac{1}{\nu}\phi_0\left(x - (t\nu)^{\frac{1}{2}}z\right)\right)dz$$
$$= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n} \exp\left\{-\frac{|z|^2}{2} - \frac{1}{\nu}\phi_0\left[(t\nu)^{\frac{1}{2}}(\xi - z)\right]\right\}dz.$$

Now, using (56) we get

$$\theta^\nu(x,t) \approx \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n}\exp\left\{-\frac{|z|^2}{2}-\frac{1}{\nu}\sum_1^n \phi_0^i\Big[(t\nu)^{\frac12}(\xi_i-z_i)\Big]\right\}dz_i$$
$$= \prod_i^n \frac{1}{(2\pi)^{\frac12}}\int_{-\infty}^{\infty}\exp\Big(-\frac{z_i^2}{2}-\frac{1}{\nu}\phi_0^i\Big[(t\nu)^{\frac12}(\xi_i-z_i)\Big]\Big)dz_i \qquad (61)$$

as $t\to\infty$ uniformly on bounded sets of ξ in R^n. Now take the i-th term in this product. Following the argument of Hopf [3], we get

$$\int_{-\infty}^{\infty}\exp\left\{-\frac{z_i^2}{2}-\frac{1}{\nu}\phi_0^i\Big[(t\nu)^{\frac12}(\xi_i-z_i)\Big]\right\}dz_i \approx \exp\Big(-\frac{\phi_i^+}{\nu}\Big)$$
$$\times\int_{-\infty}^{\xi_i}\exp\Big(-\frac{z_i^2}{2}\Big)dz_i+\exp\Big(-\frac{\phi_i^-}{\nu}\Big)\int_{\xi_i}^{\infty}\exp\Big(-\frac{z_i^2}{2}\Big)dz_i. \qquad (62)$$

Thus we have from (61) and (62),

$$(2\pi)^{\frac{n}{2}}\lim_{t\to\infty}\theta^\nu(\xi(\nu t)^{\frac12},t) = \prod_i^n \exp\Big(-\frac{1}{\nu}\phi_i^+\Big)\int_{-\infty}^{\xi_i}\exp\Big(-\frac{z_i^2}{2}\Big)dz_i$$
$$+\exp\Big(-\frac{1}{\nu}\phi_i^-\Big)\int_{\xi_i}^{\infty}\exp\Big(-\frac{z_i^2}{2}\Big)dz_i. \qquad (63)$$

Similarly, we get

$$(2\pi)^{\frac{n}{2}}\lim_{t\to\infty}(\nu t)^{\frac12}\theta^\nu_{x_l}(\xi(\nu t)^{\frac12},t)$$
$$=\prod_{i\neq l}\exp\Big(-\frac{1}{\nu}\phi_i^+\Big)\int_{-\infty}^{\xi_i}\exp\Big(-\frac{z_i^2}{2}\Big)dz_i+\exp\Big(-\frac{1}{\nu}\phi_i^-\Big)$$
$$\times\int_{\xi_i}^{\infty}\exp\Big(-\frac{z_i^2}{2}\Big)dz_i\Big[\exp\Big(-\frac{\phi_l^+}{\nu}\Big)-\exp\Big(-\frac{\phi_l^-}{\nu}\Big)\Big]\exp\Big(-\frac{\xi_l^2}{2}\Big). \qquad (64)$$

Because $(t/\nu)^{\frac12}U^\nu = -(\nu t)^{\frac12}\nabla_x\theta^\nu/\theta^\nu$ and θ^ν is bounded away from 0, we have, from (63) and (64),

$$\lim_{t\to\infty}(t/\nu)^{\frac12}U^\nu\Big[(\nu t)^{\frac12}\xi,t\Big] = -\lim_{t\to\infty}(\nu t)^{\frac12}\Big(\frac{\theta^\nu_{x_1}}{\theta^\nu},\frac{\theta^\nu_{x_2}}{\theta^\nu},\dots,\frac{\theta^\nu_{x_n}}{\theta^\nu}\Big)$$
$$= -\Big(\frac{g_1'(\xi_1)}{g_1(\xi_1)},\dots,\frac{g_n'(\xi_n)}{g_n(\xi_n)}\Big).$$

This completes the proof.

Here, we observe that for Burgers equation; that is, when $n=1$, the parameter k_1 can be computed in terms of the mass of the initial data,

$$\nu k_1 = \phi_1^+-\phi_1^- = \int_{x_0}^{\infty}u_0(y)dy-\int_{x_0}^{-\infty}u_0(y)dy = \int_{-\infty}^{\infty}u_0(y)dy$$

and then formula (60) with $n = 1$ is exactly Hopf's [3] result. Also, we note that the limit function obtained in the result written in the (x, t) variable; namely,

$$U\ (x,t) = \Big(-\nu\Big\{\log\big[v_1(x_1,t)\big]\Big\}_{x_1},\\ -\nu\Big\{\log\big[v_2(x_2,t)\big]\Big\}_{x_2},\ldots,-\nu\Big\{\log\big[v_n(x_n,t)\big]\Big\}_{x_n}\Big)$$

where for $i = 1, 2, \ldots, n$,

$$v_i(x_i,t) = \exp\Big(-\frac{k_i}{2}\Big)\int_{-\infty}^{x_i/(\nu t)^{\frac{1}{2}}} \exp\Big(-\frac{z_i^2}{2}\Big)dz_i\\ + \exp\Big(\frac{k_i}{2}\Big)\int_{x_i/(\nu t)^{\frac{1}{2}}}^{\infty} \exp\Big(-\frac{z_i^2}{2}\Big)dz_i$$

is an exact solution of (44).

3.2. N-wave solution of vector Burgers equation

We start with the solution

$$\theta(x,t) = \frac{1}{c_0} + t^{-\frac{n}{2}}\exp\Big(-\frac{|x|^2}{2t\nu}\Big)$$

of the heat equation (51) where c_0 is a constant. Its space gradient is

$$\nabla_x\theta(x,t) = -\frac{1}{\nu}\frac{x}{t}t^{-\frac{n}{2}}\exp\Big(-\frac{|x|^2}{2t\nu}\Big).$$

Consider the function $U^\infty = -\nu\nabla\theta/\theta$. By earlier discussion, this is an exact solution of the vector Burgers equation (44). On simplification, we get

$$U^\infty(x,t) = \frac{\frac{x}{t^{1/2}}}{t^{1/2}\Big[1+\frac{t^{\frac{n}{2}}}{c_0}\exp\Big(\frac{|x|^2}{2t\nu}\Big)\Big]}. \tag{65}$$

For $n = 1$, this exact solution of the Burgers equation was discovered by [5]. Sachdev et al. [6] showed that it can be obtained as time asymptotic of a pure initial value problem. This solution is called the N-wave solution. Here, we generalize it for the vector equation (44).

We choose special initial data for (44) whose mass is zero and that is antisymmetric with respect to the origin. To solve the problem explicitly, we need these data to be written as gradients. To construct such initial data, we consider the ball in R^n of radius l_0 with center 0; namely, $B(0, l_0) = [x : |x| \leq l_0]$. We take the initial condition as

$$U(x,0) = x\chi_{[|x|\leq l_0]}(x) \tag{66}$$

where $\chi_{[|x|\le l_0]}(x)$ is the characteristic function of the set $[x : |x| \le l_0]$. Note that this initial data can be written as the gradient

$$\nabla\left[\frac{|x|^2}{2}\chi_{[|x|\le l_0]}(x) + \frac{{l_0}^2}{2}(1-\chi_{[|x|\le l_0]}(x))\right].$$

So the earlier analysis holds, and we get the following formula for the solution of (44) and (66):

$$U(x,t) = -\nu\frac{\nabla Q}{Q} \tag{67}$$

where Q is given by

$$\begin{aligned} Q(x,t) = &\frac{1}{(2\pi\nu t)^{n/2}}\int_{[|y|\le l_0]}\exp\left(-\frac{1}{2\nu}\left[|y|^2 + \frac{|x-y|^2}{t}\right]\right)dy \\ &+\frac{1}{(2\pi\nu t)^{n/2}}\exp\left(-\frac{l_0^2}{2\nu}\right)\int_{[|y|>l_0]}\exp\left(-\frac{|x-y|^2}{2\nu t}\right)dy. \end{aligned}\tag{68}$$

We shall prove the following result.

Let $U(x,t)$ be the solution of (44) and (66) as given by (67) and (68), then we have

$$\lim_{t\to\infty} U(x,t) = U^{\infty}(x,t)$$

uniformly in the variable $\xi = x/(2t\nu)^{\frac{1}{2}}$ belonging to a bounded subset of R^n, where U^{∞} is given by (65) with

$$c_0 = \frac{\exp\left(\frac{l_0^2}{2\nu}\right)}{(2\pi\nu)^{\frac{n}{2}}}\left[\int_{[|y|\le l_0]}\exp\left(-\frac{|z|^2}{2\nu}\right)dz - \exp\left(-\frac{l_0^2}{2\nu}\right)|B(0,l_0)|\right]. \tag{69}$$

Here, $|B(0,l_0)|$ denotes the volume, if space dimension $n \ge 3$, area if $n = 2$, and length if $n = 1$, of $B(0,l_0)$.

To prove this result, first we note that $Q(x,t)$ can be written as

$$Q(x,t) = I_1 + \exp\left(-\frac{l_0^2}{2\nu}\right)I_2 \tag{70}$$

where

$$I_1 = \frac{1}{(2\pi\nu t)^{\frac{n}{2}}}\int_{[|y|\le l_0]}\exp\left(-\frac{1}{2\nu}\left[|y|^2 + \frac{|x-y|^2}{t}\right]\right)dy,$$

$$I_2 = \frac{1}{(2\pi\nu t)^{\frac{n}{2}}}\int_{[|y|>l_0]}\exp\left(-\frac{|x-y|^2}{2\nu t}\right)dy.$$

It can be easily checked by making use of error function and its asymptotics that, as $t \to \infty$,

$$I_1 \approx \frac{\exp\left(-\frac{|x|^2}{2\nu t}\right)}{(2\pi\nu t)^{\frac{n}{2}}} \int_{[|z|\le l_0]} \exp\left(-\frac{z^2}{2\nu}\right) dz, \; I_2 \approx \left[1 - \frac{\exp\left(-\frac{|x|^2}{2\nu t}\right)}{(2\pi\nu t)^{\frac{n}{2}}} |B(0, l_0)|\right]. \quad (71)$$

Substituting these asymptotics in (70), we get

$$Q(x, t) \approx \exp\left(-\frac{l_0^2}{2\nu}\right) + \frac{\exp\left(-\frac{|x|^2}{2\nu t}\right)}{(2\pi\nu t)^{\frac{n}{2}}} \times \left[\int_{[|z|\le l_0]} \exp\left(-\frac{|z|^2}{2\nu}\right) dz - \exp\left(-\frac{l_0^2}{2\nu}\right)|B(0, l_0)|\right]. \quad (72)$$

Similarly,

$$\nabla_x Q(x, t) \approx -\frac{1}{\nu}\frac{x}{t^{\frac{1}{2}}}\frac{1}{t^{\frac{1}{2}}}\frac{\exp\left(-\frac{|x|^2}{2\nu t}\right)}{(2\pi\nu t)^{\frac{1}{2}}} \times \left[\int_{[|z|\le l_0]} \exp\left(-\frac{|z|^2}{2\nu}\right) dz - \exp\left(-\frac{l_0^2}{2\nu}\right)|B(0, l_0)|\right]. \quad (73)$$

Using (72) and (73) in (67), we get

$$U(x, t) \approx \frac{x}{t^{\frac{1}{2}}} \cdot \frac{1}{t^{\frac{1}{2}}} \times \frac{\frac{\exp\left(-\frac{|x|^2}{2\nu t}\right)}{(2\pi\nu t)^{\frac{n}{2}}}\left[\int_{[|z|\le l_0]} \exp\left(-\frac{|z|^2}{2\nu}\right) dz - \exp\left(-\frac{l_0^2}{2\nu}\right)|B(0, l_0)|\right]}{\exp\left(-\frac{l_0^2}{2\nu}\right) + \frac{\exp\left(-\frac{|x|^2}{2\nu t}\right)}{(2\pi\nu t)^{\frac{n}{2}}}\left[\int_{[|z|\le l_0]} \exp\left(-\frac{|z|^2}{2\nu}\right) dz - \exp\left(-\frac{l_0^2}{2\nu}\right)|B(0, l_0)|\right]}.$$

On rearranging the terms, we get

$$U(x, t) \approx \frac{x/t^{\frac{1}{2}}}{t^{\frac{1}{2}}\left[1 + \frac{t^{\frac{n}{2}}}{c_0}\exp\left(\frac{|x|^2}{2t\nu}\right)\right]}$$

where c_0 is given by (69). This completes the proof.

3.3. Study of the limit $\nu \to 0$

First, we remark that the following analysis gives an explicit formula for ϕ^ν, the solution of

$$\phi_t \frac{1}{2}|\nabla\phi|^2 = \frac{\nu}{2}\Delta\phi$$
$$\phi(x, 0) = \phi_0(x); \quad (74)$$

namely,

$$\phi^\nu(x,t) = -\nu \log\left[\frac{1}{(2\pi\nu t)^{\frac{1}{2}}} \int_{R^n} \exp\left(-\frac{|x-y|^2}{2\nu t} - \frac{\phi_0(y)}{\nu}\right) dy\right]. \tag{75}$$

Following the analysis of Hopf [3] and Lax [14], see also Joseph [8], we get an explicit formula for the solution of the initial value problem for the Hamilton–Jacobi equation

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 = 0 \tag{76}$$

$$\phi(x,0) = \phi_0(x); \tag{77}$$

namely,

$$\phi(x,t) = \lim_{\nu\to 0} \phi^\nu(x,t) = \min_y\left[\phi_0(y) + \frac{1}{2t}|x-y|^2\right]. \tag{78}$$

Note that this is the explicit formula derived for the viscosity solution of (77) derived by other methods, see [15]. Furthermore, it was shown by [15] that $\phi(x,t)$ is Lipschitz continuous when $\phi_0(x)$ is and for almost every (x,t) there is a unique minimizer for (78), which we call $y_0(x,t)$. Now, to study the limit $\lim_{\nu\to 0} U^\nu(x,t)$ we note that (54) can be written as

$$U^\nu(x,t) = \int_{R^n} \left(\frac{x-y}{t}\right) d\mu^\nu_{(x,t)}(y) \tag{79}$$

where, for each (x,t) and ν, $d\mu^\nu_{(x,t)}(y)$ is a probability measure given explicitly by

$$d\mu^\nu_{(x,t)}(y) = \frac{\exp\left(-\frac{|x-y|^2}{2\nu t} - \frac{\phi_0(y)}{\nu}\right) dy}{\int_{R^n} \exp\left(-\frac{|x-y|^2}{2\nu t} - \frac{\phi_0(y)}{\nu}\right) dy}. \tag{80}$$

Following the argument of Hopf [3] and Lax [14], it can easily be seen that this measure tends to the δ-measure concentrated at $y_0(x,t)$, the minimizer of (78), which is unique for almost every (x,t). So, for almost all (x,t), we get from (79) and (80)

$$\lim_{\nu\to 0} U^\nu(x,t) = \frac{[x - y_0(x,t)]}{t} \tag{81}$$

where $y_0(x,t)$ is as before a minimizer of (78).

3.4. A boundary value problem

The method described before can be used to solve some boundary value problems as well. Let Ω be a bounded connected open set with a smooth boundary. Consider the cylindrical domain $D = \Omega \times [0, \infty)$. Consider the problem

$$
\begin{aligned}
&U_t + U \cdot \nabla U = \frac{\nu}{2} \Delta U \\
&U(x, 0) = \nabla \phi_0(x),\ x \in \Omega \\
&n(x, t) \cdot U(x, t)|_{\partial\Omega \times [0,\infty)} = \alpha. \qquad (82)
\end{aligned}
$$

Here, ϕ_0 is a smooth function from Ω to R^1, α is real constant, and $n(x, t)$ is the unit outward normal of the boundary points of D. We note that $n(x, t) \cdot U(x, t)$ is the normal component of $U(x, t)$ at the boundary point (x, t) and when $\alpha = 0$, (82) says that $U(x, t)$ is tangential to the boundary point (x, t). Here again, we seek a solution of the form $U(x, t) = \nabla \phi(x, t)$ and, as before, we get

$$
U(x, t) = -\nu \frac{\nabla \theta}{\theta} \qquad (83)
$$

where θ satisfy the linear problem

$$
\begin{aligned}
&\theta_t = \frac{\nu}{2} \Delta \theta \\
&\theta(x, 0) = \exp\left(-\frac{\phi_0(x)}{\nu}\right) \\
&\nu \partial_n \theta + \alpha \theta | \partial\Omega \times [0, \infty) = 0. \qquad (84)
\end{aligned}
$$

The explicit solution of (84) is

$$
\theta(x, t) = \sum_0^\infty c_n \exp(-\lambda_n t) \cdot \phi_n(x), \qquad (85)
$$

where

$$
c_n = \int_\Omega \exp\left(-\frac{\phi_0(x)}{\nu}\right) \phi_n(x) dx,
$$

and λ_n and ϕ_n are eigenvalues and normalized eigenfunctions of the eigenvalue problem in Ω:

$$
\begin{aligned}
&-\frac{\nu}{2} \Delta \phi = \lambda \phi, \\
&\nu \partial_n \phi + \alpha \phi|_{\partial\Omega} = 0.
\end{aligned}
$$

Using (85) in (83), we get

$$U(x, t) = -\nu \frac{\sum_1^\infty c_n \exp(-\lambda_n t) \nabla \phi_n(x)}{\sum_1^\infty c_n \exp(-\lambda_n t) \phi_n(x)}. \tag{86}$$

Letting $t \to \infty$ in (86) we get

$$\lim_{t \to \infty} U(x, t) = -\nu \frac{\nabla \phi_1}{\phi_1}.$$

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