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Exact N-wave solutions for the non-planar Burgers equation

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An exact representation of N-wave solutions for the non-planar Burgers equation

$$u_t + uu_x + \tfrac{1}{2}ju/t = \tfrac{1}{2}\delta u_{xx},$$

$j = m/n$, $m < 2n$, where m and n are positive integers with no common factors, is given. This solution is asymptotic to the inviscid solution for $|x| < \surd(2Q_0 t)$, where Q_0 is a function of the initial lobe area, as lobe Reynolds number tends to infinity, and is also asymptotic to the old age linear solution, as t tends to infinity; the formulae for the lobe Reynolds numbers are shown to have the correct behaviour in these limits. The general results apply to all $j = m/n$, $m < 2n$, and are rather involved; explicit results are written out for $j = 0, 1, \frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. The case of spherical symmetry $j = 2$ is found to be 'singular' and the general approach set forth here does not work; an alternative approach for this case gives the large time behaviour in two different time regimes. The results of this study are compared with those of Crighton & Scott (1979).

1. Introduction

One of the most fascinating partial differential equations (PDE) in the description of nonlinear phenomena is the Burgers equation

$$u_t + uu_x = \tfrac{1}{2}\delta u_{xx} \tag{1.1}$$

(see Lighthill (1956), Crighton (1979), Sachdev (1987) for a detailed physical interpretation and important solutions). The well-known Hopf–Cole transformation takes (1.1) exactly to the heat equation, and hence the solution to an initial value problem for the former can be explicitly obtained. In physical applications, however, the model equations often happen to be more complicated than (1.1), involving as they do either a geometrical expansion term, a nonlinear damping term, a variable coefficient on the right-hand side of (1.1), or a more general convection term, as in the so-called modified Burgers equation with u^2u_x replacing uu_x. The generalized Burgers equation (GBE)

$$u_t + uu_x + \tfrac{1}{2}\mathrm{d}/\mathrm{d}t[\ln A(t)]\,u = \tfrac{1}{2}\delta u_{xx} \tag{1.2}$$

describes the propagation of weakly nonlinear longitudinal waves in a gas or liquid, subject not only to the diffusion effects associated with viscosity and thermal conductivity represented by the term on the right-hand side of (1.2) but also the geometrical effects of change of the ray tube area $A(t)$ represented by the last term

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on the left. The derivation of (1.2) is sketched by Lighthill (1956) and Leibovich & Seebass (1974). For almost all these equations no Hopf–Cole-like transformation exists. Indeed, the work of Sachdev (1978) and Nimmo & Crighton (1982) shows that the only GBE for which a generalized Hopf–Cole transformation exists is the inhomogeneous Burgers equation with an additional term, $f(x,t)$, on the left of (1.1). Therefore, one must deal with GBEs directly, instead of seeking to linearize them. In a series of papers, Sachdev and his collaborators (Sachdev *et al.* 1986; Sachdev & Nair 1987; Sachdev *et al.* 1988) treated GBEs with reference to single hump initial conditions and introduced a new class of nonlinear ordinary differential equations (ODE), which they called Euler–Painlevé transcendents.

In this paper, we consider (1.2), where the ray tube area $A(t)$ is of the form $A(t) = A_0 t^j$, j being a positive constant, with N-wave initial conditions. In this case (1.2) takes the form

$$u_t + uu_x + \tfrac{1}{2}ju/t = \tfrac{1}{2}\delta u_{xx}. \tag{1.3}$$

In this context, we refer to an attempt by Sachdev *et al.* (1986) to solve this problem both analytically and numerically for spherical ($j = 2$) and cylindrical ($j = 1$) Burgers equations. This study was largely numerical, except for the cylindrically symmetric case $j = 1$. By a curious matching of the exact inviscid solution of (1.3) and the exact solution of the linearized form of the same, an exact representation of the solution of the cylindrical Burgers equation was found. It turns out that this approach can be used for general $j = m/n$, $m < 2n$ in (1.3), where m and n are positive integers and have no common factors ($m = 1$, $n = 1$ for $j = 1$ and $m = 2$, $n = 1$ for $j = 2$; $m > n$ for super-cylindrical area change and $m < n$ for the sub-cylindrical area change). We now summarize this approach. It is easy to verify that the inviscid form of (1.3) has the exact N-wave solution (3.3). It also has an antisymmetric exact solution (3.5) if the nonlinear term uu_x is dropped. The basic idea is to find an exact form which embeds these two forms: it should tend to the former for large initial Reynolds number and to the latter as $t \to \infty$. In fact, this idea was implicit in an earlier study of Sachdev & Seebass (1973) and is also supported by the singular perturbation analysis of Crighton & Scott (1979) who showed that the inviscid solution is the right outer solution, correct to all orders in a matched asymptotic expansion. A more detailed motivation of the present approach is given in §2, where we adopt it to recover the well-known exact N-wave solution for the plane Burgers equation. We first peel off most of the inviscid behaviour from the solution and introduce a reciprocal solution function. An infinite series solution for this function, in powers of $\eta = x^2/2t\delta$, the similarity variable, with coefficients depending on $\tau = t^{1/2n}$, n an integer (recall that $j = m/n$), is sought such that the solution goes to the exact linear N-wave solution as $t \to \infty$. The substitution of the series in the equation for the reciprocal function leads to an infinite system of coupled nonlinear ODEs, which is solved exactly for the unknown functions of τ. The solution ultimately involves a large system of coupled nonlinear algebraic equations. These algebraic equations can be handled with reasonable effort for $m+n \leqslant 5$; for larger values of $m+n$, this effort mounts rapidly. In fact when $m+n \leqslant 5$, by using an appropriate scaling, the coupled nonlinear algebraic equations can be reduced to an algebraic equation of finite degree in a single variable (see §4). In some cases, the algebraic system admits either complex roots or more than one real root. The former may be ruled out as physically inadmissible; the unique choice of the relevant real root may have to be made by reference to the numerical solution.

The present approach fails for the spherical case $j = 2$, for which the exact inviscid solution involves a logarithmic term: $u = x/(t\ln t)$. For this case, we give in §6 some approximate asymptotic solutions holding in various time régimes, and compare them with the matched asymptotic solution of Crighton & Scott (1979).

The present solution is an exact representation of the N-wave solution for (1.3) in the sense that the solution (2.5) is for the planar Burgers equation (2.1). It does not hold for some early time when the embryonic shock settles down to a certain steady state where the viscous and geometric terms come to a certain balance with the convective term. The solution (3.6) involves one arbitrary constant, just as the planar solution (2.5) does. The arbitrary constant (which we may refer to as the old age constant) is a function of two independent parameters, the initial Reynolds number and a spreading parameter denoted by j in this paper (see also equations (2.2)–(2.5) and (3.1)–(3.2) of Crighton & Scott (1979)). The explicit solution, for each old age constant, would itself arise from very specific, usually singular and complicated initial conditions, (see Whitham 1974 for the planar Burgers equation) but is an intermediate asymptotic, which the solution of a class of initial value problems would approach, as time becomes large (see Barenblatt 1979).

For related studies, we have already referred to the important matched asymptotic expansion approach of Crighton and his collaborators (Crighton & Scott 1979; Lee-Bapty & Crighton 1987; Nimmo & Crighton 1986), which we discuss in greater detail in §7.

The scheme of this paper is as follows: Section 2 deals with the exact solution of the plane Burgers equation; it brings out clearly, by reference to the planar N-wave, the motivation for the present approach. Section 3 gives the solution of the N-wave for general $j(=m/n)$. Explicit solutions for some special values of $j = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ are presented in §4. Explicit forms for the lobe Reynolds number $R_j(t)$ for $j = 0, 1, \frac{1}{2}, \frac{1}{3}$ are given in §5. Section 6 deals with the singular case $j = 2$. Section 7 sets forth the conclusions of this study.

2. Planar N-waves

We consider the planar Burgers equation with N-wave initial conditions

$$u_t + uu_x = \tfrac{1}{2}\delta u_{xx}, \tag{2.1}$$

$$u(x,0) = \begin{cases} x & \text{if} \quad |x| < l_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The well known Hopf–Cole transformation can be used to find explicitly the solution of (2.1) and (2.2) and hence its asymptotic form for large t. In fact

$$u(x,t) = \frac{(2\delta)^{\frac{1}{2}}\xi}{t^{\frac{1}{2}}(1+t^{\frac{1}{2}}/C_0\,\mathrm{e}^{\xi^2})} + O(1/t), \tag{2.3}$$

where
$$C_0 = \frac{2}{\pi^{\frac{1}{2}}}\mathrm{e}^{l_0^2/2\delta}\int_0^{l_0/(2\delta)^{\frac{1}{2}}} \mathrm{e}^{-z^2}\,\mathrm{d}z - (2/\pi\delta)^{\frac{1}{2}} l_0 \tag{2.4}$$

and $\xi = x/(2t\delta)^{\frac{1}{2}}$. This estimate is uniform in ξ as t becomes larger. It is easily checked that the first term in (2.3), namely

$$u^{\infty}(x,t) = \frac{x/t^{\frac{1}{2}}}{t^{\frac{1}{2}}(1+t^{\frac{1}{2}}\,\mathrm{e}^{x^2/2t\delta}/C_0)}, \tag{2.5}$$

is an exact solution of (2.1). Let

$$Q(t) = \int_0^\infty u^\infty(x,t)\,\mathrm{d}x. \tag{2.6}$$

The lobe Reynolds number for (2.5) at time t is

$$\begin{aligned} R(t) = \frac{1}{\delta}Q(t) &= \frac{1}{\delta}\int_0^\infty u^\infty(x,t)\,\mathrm{d}x \\ &= -\ln\left[1+\frac{C_0}{t^{\frac{1}{2}}}\mathrm{e}^{-x^2/2t\delta}\right]\Bigg|_0^\infty \\ &= \ln(1+C_0/t^{\frac{1}{2}}). \end{aligned} \tag{2.7}$$

Let $$R_0 = R(t_0) = \ln(1+C_0/t_0^{\frac{1}{2}}) \quad \text{and} \quad Q_0 = Q(t_0). \tag{2.8}$$

In terms of R_0, $C_0 = (\mathrm{e}^{R_0}-1)\,t_0^{\frac{1}{2}}$, therefore (2.5) can be written as

$$u^\infty(x,t) = \frac{x}{t}\left(1+\left(\frac{t}{t_0}\right)^{\frac{1}{2}}\frac{\mathrm{e}^{x^2/2t\delta}}{\mathrm{e}^{R_0}-1}\right)^{-1}. \tag{2.9}$$

For $R_0 \gg 1$, (2.9) may be approximated by

$$u^\infty(x,t) \approx \frac{x}{t}\left[1+\left(\frac{t}{t_0}\right)^{\frac{1}{2}}\exp\left[R_0\left(\frac{x^2}{2Q_0 t}-1\right)\right]\right]^{-1}, \tag{2.10}$$

for all x and t. Now for fixed t and $R_0 \to \infty$, (2.10) gives

$$u^\infty(x,t) \approx \begin{cases} x/t, & -(2Q_0 t)^{\frac{1}{2}} < x < (2Q_0 t)^{\frac{1}{2}}, \\ 0, & |x| > \surd(2Q_0 t), \end{cases} \tag{2.11}$$

which is the exact inviscid solution for (2.1). On the other hand if we let $t \to \infty$, keeping δ fixed, we have, from (2.5),

$$u^\infty \approx C_0 \frac{x}{t^{\frac{3}{2}}}\mathrm{e}^{-x^2/2t\delta}, \tag{2.12}$$

which is the old age behaviour. Thus the solution (2.5) appropriately embeds the inviscid behaviour and the old age behaviour. This motivates our approach for general j. Let

$$a = 1/C_0, \quad T = t^{\frac{1}{2}} \quad \text{and} \quad \eta = \xi^2 = x^2/2t\delta. \tag{2.13}$$

We can rewrite (2.5) as

$$u^\infty(x,t) = (2\delta)^{\frac{1}{2}}\xi/V(\eta,T), \tag{2.14}$$

where $$V(\eta,T) = \sum_{i=0}^{\infty} f_i(T)\,\eta^i/i! \tag{2.15}$$

and $$f_0(T) = T+aT^2,$$

$$f_i(t) = aT^2 \quad \text{for all} \quad i \geqslant 1. \tag{2.16}$$

In §3, we shall seek a solution of (1.3) for $j = m/n$ which meets the rather 'unusual' requirement that it goes to the inviscid form of the solution as the Reynolds number tends to infinity and to the linear solution as time tends to infinity.

3. Exact N-wave solution for general $j = m/n$, $0 < j < 2$

In this section we give an analytic approach to the N-wave solution of the equation

$$u_t + uu_x + \tfrac{1}{2}ju/t = \tfrac{1}{2}\delta u_{xx}, \tag{3.1}$$

where $0 < j < 2$, $j = m/n$, a rational number, with $m < n$ or $m > n$; the positive integers m and n have no common divisors. The inviscid form of (3.1),

$$u_t + uu_x + \tfrac{1}{2}mu/nt = 0, \tag{3.2}$$

has an exact N-wave solution

$$u(x,t) = \frac{(2\delta)^{\frac{1}{2}}}{2n/(2n-m)} \frac{x}{(2\delta t)^{\frac{1}{2}}} \frac{1}{t^{\frac{1}{2}}}. \tag{3.3}$$

The old age (linear) form of (3.1),

$$u_t + \tfrac{1}{2}mu/nt = \tfrac{1}{2}\delta u_{xx}, \tag{3.4}$$

has the 'anti-symmetric' solution

$$u(x,t) = C\frac{x}{t^{\frac{1}{2}}} \frac{1}{t^{(2n+m)/2n}} \mathrm{e}^{-x^2/2\delta t}, \tag{3.5}$$

where C is an arbitrary constant. The basic idea for the construction of the N-wave solution for (3.1) is that the solution should have the form of the exact inviscid solution (3.3) for large initial Reynolds number and asymptotic linear form (3.5) as $t \to \infty$. Motivated by our analysis of the plane Burgers equation in §2, we seek solutions of (3.1) in the form

$$u(x,t) = (2\delta)^{\frac{1}{2}} \xi / V(\eta, T), \tag{3.6}$$

where

$$\xi = x/(2\delta t)^{\frac{1}{2}}, \quad \eta = \xi^2, \quad T = t^{\frac{1}{2}}. \tag{3.7}$$

Using (3.6) and (3.7) in (3.1) we get the partial differential equation

$$V(jV - TV_T) + (2T - V)(V - 2\eta V_\eta) + 3VV_\eta + 2\eta VV_{\eta\eta} - 4\eta V_\eta^2 = 0. \tag{3.8}$$

We seek a representation of the solution of (3.8) in the form

$$V = \sum_{i=0}^{\infty} f_i(T)\, \eta^i / i!. \tag{3.9}$$

Substituting (3.9) in (3.8) and equating the coefficients of like powers of η to zero, we get

$$3f_1 + 2T - Tf_0' + (j-1)f_0 = 0, \tag{3.10}$$

$$5f_0 f_2 + 2jf_0 f_1 - f_1^2 - T(f_0 f_1)' - 2Tf_1 = 0, \tag{3.11}$$

$$7f_0 f_3 + [2(j+1)f_0 - Tf_0' - 3f_1 - 6T]f_2 - Tf_0 f_2' + 2f_1[(j+1)f_1 - Tf_1'] = 0, \tag{3.12}$$

$$(j-1)\sum_{k=0}^{i} \frac{f_k}{k!}\frac{f_{i-k}}{(i-k)!} - T\sum_{k=0}^{i} \frac{f_k}{k!}\frac{f_{i-k}'}{(i-k)!} + 2T\frac{f_i}{i!} + 3\sum_{k=0}^{i} \frac{f_k}{k!}\frac{f_{i-k+1}}{(i-k)!} - 4T\frac{f_i}{(i-1)!}$$
$$+ 2\sum_{k=0}^{i-1} \frac{f_k}{k!}\frac{f_{i-k}}{(i-k-1)!} + 2\sum_{k=0}^{i-1} \frac{f_k}{k!}\frac{f_{i-k+1}}{(i-k-1)!} - 4\sum_{k=0}^{i-1} \frac{f_{k+1}}{k!}\frac{f_{i-k}}{(i-k-1)!} = 0 \tag{3.13}$$

for $i = 3, 4, 5, \ldots$.

The system (3.10)–(3.13) is an infinite system of coupled nonlinear differential equations. Its structure is such that once f_0, f_1 and f_2 are known, all other $f_j, j \geqslant 3$ are obtained by algebraic operations alone. We determine f_0, f_1 and f_2 from (3.10) and (3.11) by making use of the inviscid behaviour (3.3) and the old age behaviour (3.5). Set

$$\tau = T^{1/n} = t^{1/2n}. \tag{3.14}$$

In terms of τ, equations (3.10) and (3.11) become

$$3f_1 + 2\tau^n - (\tau/n)f_0' + ((m-n)/n)f_0 = 0, \tag{3.15}$$

$$5f_0 f_2 + (2m/n)f_0 f_1 - f_1^2 - (\tau/n)(f_0 f_1)' - 2\tau^n f_1 = 0. \tag{3.16}$$

The inviscid and old age solutions (3.3) and (3.5) suggest that the first term in the expansions for $f_i(\tau)$ $(i = 0, 1, 2)$ should be proportional to $t^{\frac{1}{2}} = \tau^n$, while the last term should be proportional to $t^{(2n+m)/2n} = \tau^{2n+m}$. Thus, we seek solutions for $f_i(\tau)$ in the form

$$\left.\begin{aligned} f_0(\tau) &= \tau^n \sum_{i=0}^{n+m} a_i \tau^i, \\ f_1(\tau) &= \tau^n \sum_{i=0}^{n+m} b_i \tau^i, \\ f_2(\tau) &= \tau^n \sum_{i=0}^{n+m} c_i \tau^i. \end{aligned}\right\} \tag{3.17}$$

Substituting (3.17) in (3.15) and (3.16) and equating coefficients of like powers of τ to zero we obtain the following relations amongst the coefficients a_i, b_i, c_i appearing in the functions f_0, f_1 and f_2:

$$\left.\begin{aligned} &a_0 = 2n/(2n-m), \quad b_0 = 0, \quad c_0 = 0, \\ &a_{n+m} = b_{n+m} = c_{n+m}, \\ &b_i = [(2n+i-m)/3n]a_i, \quad i = 1, 2, \ldots, (n+m), \end{aligned}\right\} \tag{3.18}$$

and

$$5\alpha_i + [(2m-2n-i)/n]\beta_i - 2b_i - \gamma_i = 0, \quad i = 1, 2, \ldots, (n+m), \tag{3.19}$$

$$5\alpha_i + [(2m-2n-i)/n]\beta_i - \gamma_i = 0, \quad i = n+m+1, \ldots, 2(n+m)-1, \tag{3.20}$$

where α_i, β_i and γ_i are defined by

$$\alpha_i = \begin{cases} \sum_{q=0}^{i} a_q c_{i-q}, & i = 0, 1, \ldots, (n+m), \\ \sum_{q=i-(n+m)}^{n+m} a_q c_{i-q}, & i = (n+m+1), \ldots, 2(n+m), \end{cases} \tag{3.21}$$

$$\beta_i = \begin{cases} \sum_{q=0}^{i} a_q b_{i-q}, & i = 0, 1, \ldots, (n+m), \\ \sum_{q=i-(n+m)}^{n+m} a_q b_{i-q}, & i = (n+m+1), \ldots, 2(n+m), \end{cases} \tag{3.22}$$

$$\gamma_i = \begin{cases} \sum_{q=0}^{i} b_q b_{i-q}, & i = 0, 1, \ldots, (n+m), \\ \sum_{q=i-(n+m)}^{n+m} b_q b_{i-q}, & i = (n+m+1), \ldots, 2(n+m). \end{cases} \tag{3.23}$$

Using the expressions for b_i as given by (3.18) in (3.20), we get the following linear algebraic equations for $c_1, c_2, \ldots, c_{n+m-1}$:

$$5 \sum_{q=i-(n+m)}^{n+m} a_q c_{i-q} + \left(\frac{2m-2n-i}{n}\right) \sum_{q=i-(n+m)}^{n+m} \left(\frac{2n+i-q-m}{3n}\right) a_q a_{i-q}$$
$$- \sum_{q=i-(n+m)}^{n+m} \left(\frac{2n+i-q-m}{3n}\right)\left(\frac{2n+q-m}{3n}\right) a_q a_{i-q} = 0,$$
$$i = n+m+1, \ldots, 2(n+m)-1. \quad (3.24)_i$$

We begin with equation $(3.24)_i$ for $i = 2(n+m)-1$. Since we know that $c_{n+m} = a_{n+m}$, we can solve for c_{n+m-1} in terms of a_i. Then from the equation $(3.24)_{2(n+m)-2}$, we can solve for c_{n+m-2}. Proceeding in this manner, we can solve for $c_{n+m-1}, c_{n+m-2}, \ldots, c_1$ in terms of $a_i, i = 1, 2, \ldots, (n+m)$. Substituting these relations in (3.19) we get $(n+m)$ algebraic equations for the $(n+m-1)$ unknowns $a_1, a_2, \ldots, a_{n+m-1}$; $(a_{n+m})^{-1}$ is old age constant and must be obtained numerically.

By construction, $f_i/\tau^{2n+m} \to a_{n+m}$ as $\tau \to \infty$, for $i = 0, 1, 2$. From (3.12) and (3.13) it can be shown by induction that $f_i/\tau^{2n+m} \to a_{n+m}$ as $\tau \to \infty$, for $i \geqslant 3$. This confirms that the solution (3.6) we have constructed has the correct old age behaviour (3.5) with $C = (a_{n+m})^{-1}$.

The lobe Reynolds number for the N-wave solution $u(x,t)$ of (3.1) is defined as

$$R(t) = \frac{1}{\delta}\int_0^\infty u(x,t)\,\mathrm{d}x. \quad (3.25)$$

Integrating (3.1) with respect to x from 0 to ∞ and using (3.6), (3.9), (3.14) and (3.17), we get

$$\frac{\mathrm{d}R}{\mathrm{d}t} + \frac{mR}{2nt} = -\frac{1}{2t}\frac{1}{\sum_{i=0}^{n+m} a_i t^{\frac{1}{2}i/n}}. \quad (3.26)$$

Multiplying (3.26) by $t^{\frac{1}{2}m/n}$ and integrating the resulting expression from t to ∞, we obtain

$$R(t) = \frac{1}{2t^{\frac{1}{2}m/n}}\int_t^\infty \frac{\mathrm{d}S}{S^{(2n-m)/2n}\left[\sum_{i=0}^{n+m} a_i S^{\frac{1}{2}i/n}\right]}. \quad (3.27)$$

Notice that unlike in the work of Sachdev *et al.* (1986), we have imposed the boundary condition $R(t=\infty) = 0$; in fact, (3.27) requires that $Rt^{\frac{1}{2}m/n} \to 0$ as $t \to \infty$.

4. Explicit solution for $j = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

Since the calculations in §3 for general j are rather involved, we illustrate them by choosing some special values of j. For these values of j the solution can be obtained explicitly, although the details become more complicated as $n+m$ increases; recall that $j = m/n$.

(a) $j = 1$

This case was analysed by Sachdev *et al.* (1986) and we give here the results of their study. The constants a_i, b_i, c_i in f_0, f_1 and f_2 (see (3.17)) are given by

$$a_0 = 2, \quad a_1 = \pm 3a_2^{\frac{1}{2}},$$
$$b_0 = 0, \quad b_1 = \pm 2a_2^{\frac{1}{2}}, \quad b_2 = a_2,$$
$$c_0 = 0, \quad c_1 = \pm \tfrac{4}{5}a_2^{\frac{1}{2}}, \quad c_2 = a_2.$$

Sachev *et al.* (1986) ruled out the negative root by reference to the numerical solution. Thus f_0, f_1 and f_2 can be represented in terms of the old age constant $(a_2)^{-1}$:

$$f_0(t) = 2t^{\frac{1}{2}} + 3a_2^{\frac{1}{2}}t + a_2 t^{\frac{3}{2}},$$
$$f_1(t) = 2a_2^{\frac{1}{2}}t + a_2 t^{\frac{3}{2}},$$
$$f_2(t) = \tfrac{4}{5}a_2^{\frac{1}{2}}t + a_2 t^{\frac{3}{2}},$$

and hence all $f_i(t)$, $i \geqslant 3$, can be found from (3.12) and (3.13). The asymptotic form of the solutions as $t \to \infty$, coincides with the linear solution (3.5).

(*b*) $j = \frac{1}{2}$

In this case

$$\tau = T^{\frac{1}{2}} = t^{\frac{1}{4}}$$

and (3.15) and (3.16) become

$$3f_1 - \tfrac{1}{2}f_0 - \tfrac{1}{2}\tau f_0' + 2\tau^2 = 0, \tag{4.1}$$
$$5f_0 f_2 + f_0 f_1 - f_1^2 - 2\tau^2 f_1 - \tfrac{1}{2}\tau(f_0 f_1)' = 0. \tag{4.2}$$

We seek f_0, f_1 and f_2 in the form

$$\left.\begin{aligned} f_0 &= \tau^2[a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3], \\ f_1 &= \tau^2[b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3], \\ f_2 &= \tau^2[c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3], \end{aligned}\right\} \tag{4.3}$$

where

$$a_0 = \tfrac{4}{3} \tag{4.4}$$

(see (3.14), (3.17) and (3.18)).

Substituting (4.3) in (4.1) and equating coefficients of like powers of τ to zero we get

$$b_0 = 0, \quad b_1 = \tfrac{2}{3}a_1, \quad b_2 = \tfrac{5}{6}a_2, \quad b_3 = a_3. \tag{4.5}$$

Substituting (4.3) in (4.2) and equating coefficients of same powers of τ to zero, we get

$$c_0 = 0, \quad c_1 = \tfrac{2}{5}a_1, \quad c_2 = -\tfrac{1}{30}a_1^2 + \tfrac{7}{12}a_2, \quad c_3 = a_3 \tag{4.6}$$

and a_1, a_2 and a_3 satisfy

$$\left.\begin{aligned} a_1^3 - 6a_3 &= 0, \\ a_1^2 + a_2 &= 0, \\ a_3 + \tfrac{1}{24}a_1 a_2 - \tfrac{1}{8}a_1^3 &= 0. \end{aligned}\right\} \tag{4.7}$$

Here a_3^{-1} is the old age constant to be determined numerically. The system of three equations (4.7) relates two unknowns a_1 and a_2. The only real solution of the first of (4.7) is

$$a_1 = 6^{\frac{1}{3}}a_3^{\frac{1}{3}}. \tag{4.8}$$

From the second equation of (4.7), we get

$$a_2 = -6^{\frac{2}{3}}a_3^{\frac{2}{3}}. \tag{4.9}$$

It is easy to check that the last equation of (4.7) is identically satisfied by (4.8) and (4.9). Thus we have

$$\left.\begin{aligned} &a_0 = \tfrac{4}{3}, \quad a_1 = 6^{\frac{1}{3}}a_3^{\frac{1}{3}}, \quad a_2 = -6^{\frac{2}{3}}a_3^{\frac{2}{3}} \\ &b_0 = 0, \quad b_1 = \tfrac{2}{3}6^{\frac{1}{3}}a_3^{\frac{1}{3}}, \quad b_2 = -\tfrac{5}{6}6^{\frac{2}{3}}a_3^{\frac{2}{3}}, \quad b_3 = a_3 \\ &c_0 = 0, \quad c_1 = \tfrac{2}{5}6^{\frac{1}{3}}a_3^{\frac{1}{3}}, \quad c_2 = -\tfrac{37}{60}6^{\frac{2}{3}}a_3^{\frac{2}{3}}, \quad c_3 = a_3. \end{aligned}\right\} \tag{4.10}$$

(*c*) $j = \frac{1}{3}$

Here we set $\tau = T^{\frac{1}{3}} = t^{\frac{1}{6}}$, so (3.15) and (3.16) become

$$3f_1 - \tfrac{2}{3}f_0 - \tfrac{1}{3}\tau f_0' + 2\tau^3 = 0, \tag{4.11}$$

$$5f_0 f_2 + \tfrac{2}{3}f_0 f_1 - f_1^2 - 2\tau^3 f_1 - \tfrac{1}{3}\tau(f_0 f_1)' = 0. \tag{4.12}$$

In view of (3.17), for $j = \frac{1}{3}$, we seek f_0, f_1 and f_2 in the form

$$\left.\begin{aligned} f_0 &= \tau^3[a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4], \\ f_1 &= \tau^3[b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4], \\ f_2 &= \tau^3[c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3 + c_4\tau^4], \end{aligned}\right\} \tag{4.13}$$

where

$$a_0 = \tfrac{6}{5}. \tag{4.14}$$

Substituting (4.13) in (4.11), equating coefficients of different powers of τ to zero and using (4.14), we get

$$b_0 = 0, \quad b_1 = \tfrac{2}{3}a_1, \quad b_2 = \tfrac{7}{9}a_2, \quad b_3 = \tfrac{8}{9}a_3, \quad b_4 = a_4. \tag{4.15}$$

Similarly from (4.13) and (4.12) we obtain

$$c_0 = 0, \quad c_1 = \tfrac{4}{9}a_1, \quad c_2 = \tfrac{77}{135}a_2 - \tfrac{2}{27}a_1^3, \quad c_3 = \tfrac{20}{27}a_3, \quad c_4 = a_4, \tag{4.16}$$

where a_1, a_2, a_3 and a_4 are related by the coupled system of nonlinear algebraic equations

$$\left.\begin{aligned} a_4 + \tfrac{20}{27}a_1 a_3 - \tfrac{25}{54}a_1^2 a_2 + \tfrac{35}{162}a_2^2 &= 0, \\ a_1 a_4 - \tfrac{5}{12}a_1^3 a_3 + \tfrac{7}{36}a_2 a_3 &= 0, \\ a_2 a_4 - \tfrac{2}{15}a_3^2 - a_1^2 a_4 &= 0, \\ a_3 - \tfrac{25}{12}a_1^3 + \tfrac{15}{4}a_1 a_2 &= 0. \end{aligned}\right\} \tag{4.17}$$

Here a_4^{-1} is the old age constant. Thus, the system of four equations (4.17) relates the three unknowns, a_1, a_2 and a_3. Solving this algebraic system we get

$$a_1 = (\tfrac{27}{250})^{\frac{1}{4}} a_4^{\frac{1}{4}}, \quad a_2 = 5(\tfrac{27}{250})^{\frac{2}{4}} a_4^{\frac{2}{4}}, \quad a_3 = -\tfrac{50}{3}(\tfrac{27}{250})^{\frac{3}{4}} a_4^{\frac{3}{4}}. \tag{4.18}$$

From (4.14)–(4.18) we finally have

$$\left.\begin{aligned} &a_0 = \tfrac{6}{5}, \quad a_1 = (\tfrac{27}{250})^{\frac{1}{4}} a_4^{\frac{1}{4}}, \quad a_2 = 5(\tfrac{27}{250})^{\frac{2}{4}} a_4^{\frac{2}{4}}, \quad a_3 = -\tfrac{50}{3}(\tfrac{27}{250})^{\frac{3}{4}} a_4^{\frac{3}{4}}, \\ &b_0 = 0, \quad b_1 = \tfrac{2}{3}(\tfrac{27}{250})^{\frac{1}{4}} a_4^{\frac{2}{4}}, \\ &b_2 = \tfrac{35}{9}(\tfrac{27}{250})^{\frac{2}{4}} a_4^{\frac{2}{4}}, \quad b_3 = -\tfrac{400}{27}(\tfrac{27}{250})^{\frac{3}{4}} a_4^{\frac{3}{4}}, \\ &c_0 = 0, \quad c_1 = \tfrac{4}{9}(\tfrac{27}{250})^{\frac{1}{4}} a_4^{\frac{1}{4}}, \quad c_2 = \tfrac{25}{9}(\tfrac{27}{250})^{\frac{2}{4}} a_4^{\frac{2}{4}}, \\ &c_3 = -\tfrac{1000}{81}(\tfrac{27}{250})^{\frac{3}{4}} a_4^{\frac{3}{4}}, \quad c_4 = b_4 = a_4. \end{aligned}\right\} \tag{4.19}$$

(*d*) $j = \frac{1}{4}$

Pursuing the analysis which is a direct extension of that for the cases $j = 1, \frac{1}{2}$ and $\frac{1}{3}$, that we have set out in detail earlier, we were able to obtain the coefficients in f_0, f_1 and f_2 for $j = \frac{1}{4}$ as well. Here we give the final results. With $\tau = t^{\frac{1}{8}}$, f_0, f_1 and f_2 have the form

$$f_0 = \tau^4\left[\sum_{i=0}^{5} a_i \tau^i\right], \quad f_1 = \tau^4\left[\sum_{i=0}^{5} b_i \tau^i\right], \quad f_2 = \tau^4\left[\sum_{i=0}^{5} c_i \tau^i\right], \tag{4.20}$$

where
$$\left.\begin{aligned} &a_0 = \tfrac{8}{7}, \quad b_0 = 0, \quad b_1 = \tfrac{2}{3}a_1, \quad b_2 = \tfrac{3}{4}a_2, \\ &b_3 = \tfrac{5}{6}a_3, \quad b_4 = \tfrac{11}{12}a_4, \quad b_5 = a_5, \\ &c_0 = 0, \quad c_1 = \tfrac{7}{15}a_1, \quad c_2 = \tfrac{9}{16}a_2 - \tfrac{7}{72}a_1^2, \\ &c_3 = -\tfrac{161}{960}a_1 a_2 + \tfrac{49}{576}a_1^3 + \tfrac{2}{3}a_3, \quad c_4 = \tfrac{193}{240}a_4, \quad c_5 = a_5; \end{aligned}\right\} \tag{4.21}$$

a_5^{-1} is the old age constant and a_1, a_2, a_3 and a_4 are related to it by the following five nonlinear coupled algebraic equations:

$$\left.\begin{aligned} \tfrac{1}{7}a_4 + \tfrac{245}{576}a_1^4 - \tfrac{763}{576}a_1^2 a_2 + \tfrac{29}{36}a_1 a_3 + \tfrac{3}{8}a_2^2 &= 0, \\ \tfrac{4}{7}a_5 + \tfrac{7}{9}a_1 a_4 + \tfrac{13}{24}a_2 a_3 - \tfrac{35}{72}a_1^2 a_3 + \tfrac{245}{576}a_1^2 a_2 - \tfrac{161}{192}a_1 a_2^2 &= 0, \\ a_1 a_5 + \tfrac{11}{24}a_2 a_4 + \tfrac{5}{36}a_3^2 + \tfrac{245}{576}a_1^2 a_3 - \tfrac{161}{192}a_1 a_2 a_3 - \tfrac{35}{72}a_1^2 a_4 &= 0, \\ \tfrac{5}{2}a_2 a_5 + \tfrac{5}{9}a_3 a_4 + \tfrac{245}{144}a_1^3 a_4 - \tfrac{161}{48}a_1 a_2 a_4 - \tfrac{35}{18}a_1^2 a_5 &= 0, \\ a_3 a_5 - \tfrac{1}{9}a_4^4 + \tfrac{245}{144}a_1^3 a_5 - \tfrac{161}{48}a_1 a_2 a_5 &= 0. \end{aligned}\right\} \tag{4.22}$$

By an appropriate scaling we were able to reduce (4.22) to a fifth degree polynomial equation in a single variable and hence solve the latter numerically. We thus obtained

$$a_1 = 0.34906 a_5^{\frac{1}{5}}, \quad a_2 = 0.47383 a_5^{\frac{2}{5}}, \quad a_3 = 1.44718 a_5^{\frac{3}{5}}, \quad a_4 = -2.94671 a_5^{\frac{4}{5}}. \tag{4.23}$$

Using (4.23) in (4.21), we get the coefficients of f_0, f_1 and f_2 in terms of the old age constant a_5^{-1}, which itself must be determined numerically.

We have also carried out the analysis for the case $j = \frac{3}{2}$ for which $m+n = 5$, the same as for $j = \frac{1}{4}$. All the details, including the structure of the algebraic equations analogous to (4.22), are entirely similar; therefore we omit the discussion of this case for the sake of brevity.

5. The lobe Reynolds number

It is possible to write the explicit form of the lobe Reynolds number $R_j(t)$ for some special cases from (3.27). We recall that we here impose the condition $t^{j/2}R_j(t) \to 0$ as $t \to \infty$.

$$(a)\ j = 0: \quad R_0(t) = \ln\left[1 + \frac{1}{a_1 t^{\frac{1}{2}}}\right], \tag{5.1}$$

$$(b)\ j = 1: \quad R_1(t) = \frac{1}{(ta_2)^{\frac{1}{2}}} \ln\left[1 + \frac{1}{1+(ta_2)^{\frac{1}{2}}}\right], \tag{5.2}$$

$$(c)\ j = \tfrac{1}{2}: \quad R_{\frac{1}{2}}(t) = \frac{2}{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}}} \ln\left[\frac{(6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}} - 1 - \sqrt{3})^{\frac{1}{2}(\sqrt{3}+1)}}{(6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}} - 4)\,(6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}} - 1 + \sqrt{3})^{\frac{1}{2}(\sqrt{3}-1)}}\right], \tag{5.3}$$

$$(d)\ j = \tfrac{1}{3}: \quad R_{\frac{1}{3}}(t) = 81/125T$$
$$\times\left\{\ln\left[\frac{(T^2+pT+q)^{\frac{1}{2}(A+B)}}{(T-\alpha)^A\,(T-\beta)^B}\right] + \frac{(Cp-2D)}{\sqrt{(4q-p^2)}}\left[\arctan\left(\frac{p+2T}{\sqrt{(4q-p^2)}}\right) - \tfrac{1}{2}\pi\right]\right\}, \tag{5.4}$$

where $T = (27/250)^{\frac{1}{4}} a_4^{\frac{1}{4}} t^{\frac{1}{6}}$, $A = (\alpha^2-1)/\alpha\beta$, $B = (\beta^2-1)/\alpha\beta$, $C = (2-\beta^2-\alpha^2)/\alpha\beta$, $D = [(\alpha+p)(1-\alpha^2)+(\beta+p)(1-\beta^2)]/\alpha\beta$, $p = \beta - \frac{3}{5}$, $q = \beta^2 - \frac{3}{5}\beta - (9/50)$ and α and β are the real roots of the equation $\tau^4 - (9/5)\tau^3 + (27/50)\tau^2 + (27/250)\tau + (81/625) = 0$; in fact $\alpha = (6/5)$ and $\beta \approx 0.99222$.

It is curious that R_1 has a form very similar to R_0, the additional factor $1/(ta_2)^{\frac{1}{2}}$ being simply the geometrical attenuation. It is easy to conclude from these results that the Reynolds number formulae, and hence our solution, become valid earlier in time, the smaller the initial Reynolds number. This becomes evident if we write for (5.1), (5.2) and (5.3) the large-time expansions:

$$R_0(t) = \frac{1}{a_1 t^{\frac{1}{2}}} - \frac{1}{2a_1 t} + \ldots, \tag{5.5}$$

$$R_1(t) = \frac{1}{a_2 t} - \frac{3}{2a_2^{\frac{3}{2}} t^{\frac{3}{2}}} + \ldots, \tag{5.6}$$

$$R_{\frac{1}{2}}(t) = \frac{1}{a_3 t^{\frac{3}{4}}} + \frac{4}{6^{\frac{1}{3}} a_3^{\frac{4}{3}} t} + \ldots. \tag{5.7}$$

Here we note that the first terms in (5.5)–(5.7) are simply the Reynolds number obtained from the old age (linear) solution (3.5).

In the large Reynolds number régime, if the initial Reynolds number $R_j(t_0)$ is taken into account, $R_j(t)$ for $t \geqslant t_0$ is given by

$$(e)\ j = 0: \quad R_0(t) = R_0(t_0) - \ln\left[\frac{1+(1/a_1 t^{\frac{1}{2}})}{1+(1/a_1 t_0^{\frac{1}{2}})}\right]. \tag{5.8}$$

$$(f)\ j = 1: \quad R_1(t) = R_1(t_0)\left(\frac{t_0}{t}\right)^{\frac{1}{2}} - \frac{1}{(a_2 t)^{\frac{1}{2}}} \ln\left[\frac{(a_2^{\frac{1}{2}} t^{\frac{1}{2}}+1)\,(a_2^{\frac{1}{2}} t_0^{\frac{1}{2}}+2)}{(a_2^{\frac{1}{2}} t_0^{\frac{1}{2}}+1)\,(a_2^{\frac{1}{2}} t^{\frac{1}{2}}+2)}\right]. \tag{5.9}$$

$$(g)\ j = \tfrac{1}{2}: \quad R_{\frac{1}{2}}(t) = R_{\frac{1}{2}}(t_0)\left(\frac{t_0}{t}\right)^{\frac{1}{4}}$$

$$-\frac{2}{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}}} \ln\left\{\left[\frac{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}} - 4}{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t_0^{\frac{1}{4}} - 4}\right]\left[\left(\frac{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}} - 1 + \sqrt{3}}{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t_0^{\frac{1}{4}} - 1 + \sqrt{3}}\right)^{\frac{1}{2}(\sqrt{3}-1)}\right] \Big/ \left[\left(\frac{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t^{\frac{1}{4}} - 1 - \sqrt{3}}{6^{\frac{1}{3}} a_3^{\frac{1}{3}} t_0^{\frac{1}{4}} - 1 - \sqrt{3}}\right)^{\frac{1}{2}(\sqrt{3}+1)}\right]\right\}. \tag{5.10}$$

If we now expand the expressions (5.8)–(5.10) for $t \gg t_0$ and $R_j(t) \gg 1$, we obtain

$$R_0(t) \approx R_0(t_0) - \tfrac{1}{2}\ln(t_0/t),$$
$$R_1(t) \approx R_1(t_0)\,(t_0/t)^{\frac{1}{2}} - \tfrac{1}{2}(1-(t_0/t)^{\frac{1}{2}}),$$
$$R_{\frac{1}{2}}(t) \approx R_{\frac{1}{2}}(t_0)\,(t_0/t)^{\frac{1}{4}} - \tfrac{3}{2}(1-(t_0/t)^{\frac{1}{4}}).$$

These are the appropriate large Reynolds number forms, obtained earlier by Liebovich & Seebass (1974) for $j = 0, 1, 2$ on using an approximate analysis. We again emphasize that $R_j(t_0)$ does not denote the lobe Reynolds number for any (particular) known initial conditions.

6. Spherical N-wave

Now we consider the spherical Burgers equation

$$u_t + uu_x + u/t = \tfrac{1}{2}\delta u_{xx}, \tag{6.1}$$

with N-wave initial condition

$$u(x,0) = \begin{cases} x & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

We introduce the new variables, namely t and

$$\left.\begin{aligned} \eta &= xt^{\alpha}(\ln t)^{\beta}, \\ V(\eta, t) &= (t+c)^{-a}(\ln(t+c))^{-b}u, \end{aligned}\right\} \tag{6.3}$$

and write (6.1) as

$$V_t + \left[\frac{a+1}{t+c} + \frac{b}{(t+c)\ln(t+c)}\right]V + \left[\frac{\alpha}{t} + \frac{\beta}{t\ln t}\right]\eta V_\eta$$
$$+ (t+c)^a[\ln(t+c)]^b t^{\alpha}(\ln t)^{\beta} V V_\eta = \tfrac{1}{2}\delta t^{2\alpha}(\ln t)^{2\beta} V_{\eta\eta}. \tag{6.4}$$

In the limit $t \gg 1$, different balances of terms in (6.4) are possible, of which only two are relevant here. We consider each of these balances separately.

Case 1. $\alpha = 0$, $\beta = 0$, $a = -1$, $b = -1$.

In this case (6.4) has the form

$$V_t - \left[\frac{1}{(t+c)\ln(t+c)}\right]V + \frac{1}{(t+c)\ln(t+c)} V V_x = \tfrac{1}{2}\delta V_{xx} \tag{6.5}$$

and the initial data (6.2) become

$$V(x,0) = \begin{cases} (c\ln c)\,x & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

Seeking the solution of (6.5) in the form

$$V = V^0 + \frac{1}{(t+c)\ln(t+c)} V^1 + \ldots, \tag{6.7}$$

we have

$$\left.\begin{aligned} V^0_t &= \tfrac{1}{2}\delta V^0_{xx}, \\ V^0(x,0) &= \begin{cases} (c\ln c)\,x & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}\right\} \tag{6.8}$$

and

$$\left.\begin{aligned} V^1_t &= \tfrac{1}{2}\delta V^1_{xx} + V^0 - (\tfrac{1}{2}V^0)^2_x, \\ V^1(x,0) &= 0, \quad -\infty < x < \infty. \end{aligned}\right\} \tag{6.9}$$

The explicit solution of (6.8) is

$$V^0(x,t) = c\ln c \frac{1}{(2\pi t\delta)^{\frac{1}{2}}} \int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta} y\,\mathrm{d}y. \tag{6.10}$$

We try to write (6.10) in a more convenient form. For this purpose, consider

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y = -\int_{-1}^{1} \frac{\partial}{\partial y}(\mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta})\,\mathrm{d}y = \mathrm{e}^{-\frac{1}{2}(x+1)^2/t\delta} - \mathrm{e}^{-\frac{1}{2}(x-1)^2/t\delta}, \tag{6.11}$$

and
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y = \frac{1}{t\delta}\int_{-1}^{1} y\,\mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y - \frac{x}{t\delta}\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y. \tag{6.12}$$

From (6.11) and (6.12), we have

$$\int_{-1}^{1} y\,\mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y = x\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y + (t\delta)\,[\mathrm{e}^{-\frac{1}{2}(x+1)^2/t\delta} - \mathrm{e}^{-\frac{1}{2}(x-1)^2/t\delta}]. \tag{6.13}$$

Using (6.13) in (6.10), we get

$$V^0(x,t) = c\ln c\left[\frac{x}{(2\pi t\delta)^{\frac{1}{2}}}\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y + \left(\frac{t\delta}{2\pi}\right)^{\frac{1}{2}}\left(\mathrm{e}^{-\frac{1}{2}(x+1)^2/t\delta} - \mathrm{e}^{-\frac{1}{2}(x-1)^2/t\delta}\right)\right]. \tag{6.14}$$

Once $V^0(x,t)$ is known, $V^1(x,t)$ can be found from (6.9):

$$V^1(x,t) = \int_0^t\int_{-\infty}^{\infty} \frac{V^0(y,s) - (\frac{1}{2}V^0)^2_y(y,s)}{[2\pi(t-s)\,\delta]^{\frac{1}{2}}}\,\mathrm{e}^{-(x-y)^2/(2(t-s)\delta)}\,\mathrm{d}y\,\mathrm{d}s. \tag{6.15}$$

Thus we get the following expansion for $u(x,t)$:

$$u(x,t) = \frac{c\ln c}{(t+c)\ln(t+c)}\left[\frac{x}{(2\pi t\delta)^{\frac{1}{2}}}\int_{-1}^{1} \mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y + (\tfrac{1}{2}t\delta/\pi)^{\frac{1}{2}}\left(\mathrm{e}^{-\frac{1}{2}(x+1)^2/t\delta} - \mathrm{e}^{-\frac{1}{2}(x-1)^2/t\delta}\right)\right]$$
$$+\frac{c\ln c}{(t+c)^2[\ln(t+c)]^2}\int_0^t\int_{-\infty}^{\infty}\frac{V^0(y,s) - (\frac{1}{2}V^0)^2_y(y,s)}{[2\pi(t-s)\,\delta]^{\frac{1}{2}}}\,\mathrm{e}^{-\frac{1}{2}(x-y)^2/(t-s)\,\delta}\,\mathrm{d}y\,\mathrm{d}s + \dots. \tag{6.16}$$

The constant c must be chosen suitably. Since

$$\frac{1}{(2\pi t\delta)^{\frac{1}{2}}}\int_{-1}^{1}\mathrm{e}^{-\frac{1}{2}(x-y)^2/t\delta}\,\mathrm{d}y = \frac{1}{\pi^{\frac{1}{2}}}\int_{(x-1)/(2t\delta)^{\frac{1}{2}}}^{(x+1)/(2t\delta)^{\frac{1}{2}}}\mathrm{e}^{-z^2}\,\mathrm{d}z \rightarrow 1, \quad \text{as} \quad \delta\rightarrow 0$$

for $|x| < 1$, the first term of the right-hand side of (6.16) tends to $c(\ln c)\,x/[(t+c)\ln(t+c)]$ as $\delta\rightarrow 0$ provided $|x| < 1$. Now we find $c > 0$ such that $c\ln c = 1$; in fact $c \approx 1.763$. It can easily be shown that the second term of (6.16) is of the order $O(1/(t+c)(\ln(t+c))^2)$ as $\delta\rightarrow 0$. Thus (6.16) has the correct behaviour as $\delta\rightarrow 0$ for this choice of c: the first term tends to the inviscid solution which is also the outer solution correct to all orders in the matched asymptotic analysis of Crighton & Scott (1979).

Case 2. $\alpha = 0$, $\beta = 0$, $b = 0$, $a = -1$

In this case (6.4) becomes

$$V_t + \frac{1}{t+c}VV_x = \tfrac{1}{2}\delta V_{xx}. \tag{6.17}$$

The initial condition (6.2) becomes

$$V(x,0) = \begin{cases} cx & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise.}\end{cases} \tag{6.18}$$

Seeking an expansion of V in the form

$$V = V^0 + \frac{1}{t+c}V^1 + \dots, \tag{6.19}$$

we get from (6.17) and (6.18) the following systems for V^0 and V^1:

$$\left.\begin{aligned} V^0_t &= \tfrac{1}{2}\delta V^0_{xx}, \\ V^0(x,0) &= \begin{cases} cx & \text{if} \quad |x| < 1, \\ 0 & \text{otherwise,}\end{cases}\end{aligned}\right\} \tag{6.20}$$

and
$$\left.\begin{aligned} V^1_t &= \tfrac{1}{2}\delta V^1_{xx} - \tfrac{1}{2}(V^0)^2_x, \\ V^1(x,0) &= 0, \quad -\infty < x < \infty. \end{aligned}\right\} \tag{6.21}$$

As for the case 1, we obtain

$$V^0(x,t) = c\left[\frac{x}{(2\pi t\delta)^{\frac{1}{2}}}\int_{-1}^{1} e^{-\frac{1}{2}(x-y)^2/t\delta}\,dy + (\tfrac{1}{2}t\delta/\pi)^{\frac{1}{2}}(e^{-\frac{1}{2}(x+1)^2/t\delta} - e^{-\frac{1}{2}(x-1)^2/t\delta})\right] \tag{6.22}$$

and
$$V^1(x,t) = -\int_0^t\int_{-\infty}^{\infty}\frac{(\frac{1}{2}V^0)^2_y\,(y,s)}{[2\pi(t-s)\,\delta]^{\frac{1}{2}}}e^{-(x-y)^2/(2(t-s)\,\delta)}\,dy\,ds. \tag{6.23}$$

The solution for the variable $u(x,t)$ becomes

$$u(x,t) = \frac{c}{(t+c)}\left[\frac{x}{(2\pi t\delta)^{\frac{1}{2}}}\int_{-1}^{1} e^{-\frac{1}{2}(x-y)^2/t\delta}\,dy + (\tfrac{1}{2}t\delta/\pi)^{\frac{1}{2}}(e^{-\frac{1}{2}(x+1)^2/t\delta} - e^{-\frac{1}{2}(x-1)^2/t\delta})\right] + \frac{c}{(t+c)^2}V^1(x,t) + \ldots, \tag{6.24}$$

where $V^1(x,t)$ is given by (6.23). Here the constant c remains to be determined. This constant is again a complicated function of two parameters, the initial Reynolds number and a spreading parameter. For each pair of values of the parameters, one would have to solve the generalized Burgers equation exactly, follow the solution to larger times and compare the results with (6.25). We consider the first term of (6.24), calling it $u^0(x,t)$. We show that is essentially the old age asymptotic form of Crighton & Scott (1979), up to a constant, obtained by matched asymptotic expansion. For this purpose, we recall that

$$\operatorname{erfc}(y) = \frac{2}{\pi^{\frac{1}{2}}}\int_y^{\infty} e^{-z^2}\,dz$$

and note that

$$\begin{aligned}\frac{1}{(2\pi t\delta)^{\frac{1}{2}}}\int_{-1}^{1} e^{-\frac{1}{2}(x-y)^2/t\delta}\,dy &= \frac{1}{\pi^{\frac{1}{2}}}\int_{(x-1)/(2t\delta)^{\frac{1}{2}}}^{(x+1)/(2t\delta)^{\frac{1}{2}}} e^{-x^2}\,dz \\ &= \tfrac{1}{2}[\operatorname{erfc}((x-1)/(2t\delta)^{\frac{1}{2}}) - \operatorname{erfc}((x+1)/(2t\delta)^{\frac{1}{2}})].\end{aligned}$$

Hence

$$u^0(x,t) = \frac{1}{2}\frac{cx}{(t+c)}\left[\operatorname{erfc}\left(\frac{x-1}{(2t\delta)^{\frac{1}{2}}}\right) - \operatorname{erfc}\left(\frac{x+1}{(2t\delta)^{\frac{1}{2}}}\right)\right] + \frac{c}{t+c}\left(\frac{\delta t}{2\pi}\right)^{\frac{1}{2}}[e^{-(x+1)^2/(2t\delta)^{\frac{1}{2}}} - e^{-(x-1)^2/(2t\delta)^{\frac{1}{2}}}]. \tag{6.25}$$

Letting $t\to\infty$ in (6.25), we get

$$u^0(x,t) \approx \frac{2^{\frac{1}{2}}c}{3\pi^{\frac{1}{2}}\delta^{\frac{3}{2}}}\frac{x}{t^{\frac{5}{2}}}e^{-x^2/2t\delta}. \tag{6.26}$$

Using matched asymptotic expansions, Crighton & Scott (1979) obtained the old age form as

$$u^0_c(x,t) = \frac{1}{2}\frac{X}{t}[\operatorname{erfc}[\nu(X-1)] - \operatorname{erfc}[\nu(X+1)]] + \frac{1}{2\pi^{\frac{1}{2}}\nu t}[e^{-\nu^2(X+1)^2} - e^{-\nu(X-1)^2}], \tag{6.27}$$

where $X = x/T_2^{\frac{1}{2}}$, $\nu = \frac{1}{2}e^{-\frac{1}{2}T_2}t^{-\frac{1}{2}}$ and T_2 solves $\epsilon T_2^{-1}e^{T_2} = 1$, $\epsilon = \frac{1}{2}\delta$, and showed that as $t \to \infty$

$$u_c^0(x,t) \approx \frac{1}{t}\frac{1}{6\pi^{\frac{1}{2}}}T_2^{\frac{1}{2}}\frac{x}{(\epsilon t)^{\frac{3}{2}}}e^{-x^2/2t\delta}. \tag{6.28}$$

Since $\epsilon = \frac{1}{2}\delta$, we get

$$u_c^0(x,t) \approx T_2^{\frac{1}{2}}\frac{2^{\frac{1}{2}}}{3\pi^{\frac{1}{2}}\delta^{\frac{3}{2}}}\frac{x}{t^{\frac{5}{2}}}e^{-x^2/2t\delta}. \tag{6.29}$$

The asymptotic forms (6.26) and (6.29) are the same except for a constant multiple. Crighton and Scott were able to find the arbitrary constant in (6.29) by suitably matching it to solutions holding in earlier time régimes. Our solution (6.25) would have to be supplemented by numerical solution in order to identify the unknown constant.

7. Conclusions

The non-planar Burgers equation (3.1), like most generalized Burgers equations, does not admit a Hopf–Cole transformation, and therefore, must be treated directly. In the present paper we are concerned with the N-wave solutions of this equation. We attempt to mimic the planar N-wave solution (2.14) by ensuring that the non-planar N-wave solution tends, in the limit of large initial Reynolds number, with t fixed, to the inviscid solution (3.3) and, in the limit $t \to \infty$ to the old age solution (3.5). Fortunately, it becomes possible to do that for $0 < j < 2$, where $j = m/n$, m and n being positive integers with no common factors. The case $j = 2$ is exceptional since the inviscid solution $u = x/(t \ln t)$ involves a logarithmic term and our method does not work. We return to this special case presently. To obtain the desired asymptotic solution we found it convenient to peel off 'most' of the inviscid behaviour and introduce a reciprocal function (see equation 3.6). The infinite series form (3.9) of the reciprocal function leads to an infinite system of coupled ODEs for the time functions $f_i(\tau)$. These equations have a special structure, so that if the first three functions f_0, f_1 and f_2 can be obtained from the first two equations, all other $f_i (i \geqslant 3)$ can be found by algebraic operations alone. The knowledge of the inviscid form and the (linear) old age form of the solution helps us to accomplish this.

The analysis of the infinite system of ODEs leads finally to a system of coupled nonlinear algebraic equations for the coefficients $a_1, a_2, \ldots, a_{n+m-1}$, occurring in the first unknown function $f_0(t) = t^{\frac{1}{2}}\sum_{i=0}^{m+n} a_i t^{\frac{1}{2}i/n}$, which, after some non-trivial manipulation and scaling, can be reduced to an algebraic equation of finite degree in one of the unknowns. The roots of this equation help us to find all the unknown constants in the polynomial functions $f_i(t)$ in terms of the old age constant a_{n+m}^{-1}, which must remain unknown and be related to the numerical solution of a given initial value problem. The roots of the algebraic equation must be examined for each j. For $j = \frac{1}{2}, \frac{1}{3}$ there is only one real root; for $j = 1$, of the possible two roots, one was chosen after comparison with the numerical solution. The solution that we have found here for general $j = m/n$ is an exact representation of the N-wave which holds for all time except for some initial time over which all the effects – convective, geometrical and diffusive – come to a certain balance (see Leibovich & Seebass 1974).

Now we turn to the 'singular' case of spherical symmetry with $j = 2$. In this case, our general approach does not work. If there does exist a single exact representation, it must be much more complicated than (3.6)–(3.9) and must involve logarithmic terms in $f_i(t)$. However, we have adopted a different approach for this case, as explained in §6, to discover large time behaviour. We introduce new independent and dependent variables (6.3) and seek suitable 'balances' in the transformed equation to obtain simpler forms of (6.1). It turns out that there are two important balances: (*a*) which leads to a correction to the inviscid solution, $u = x/(t \ln t)$, to include linear diffusive effects (see 6.16), and (*b*) which corrects the linear diffusive solution to incorporate the effects of nonlinearity.

The approximate solution corresponding to (*a*) is fully determined in the form (6.16) while that corresponding to (*b*) is found within an arbitrary constant. Indeed the latter approximate solution in the limit $t \to \infty$ is exactly the same as found earlier by Crighton & Scott (1979) using matched asymptotic expansions; they were also able to find the arbitrary constant by matching this solution to other solutions holding in earlier time régimes. The approximate solution (6.16) is new and has the right behaviour in the early evolution of the N-wave.

Now we compare our exact analytic representation (3.6) of the N-wave solutions for general $j = m/n$ with matched asymptotic solutions of Crighton and his collaborators (Crighton & Scott 1979; Nimmo & Crighton 1986). Assuming the diffusivity to be small (which is a physically correct assumption), Crighton and his collaborators have solved the N-wave problem for $j = 1, 2$, using matched asymptotic expansions. Crighton & Scott (1979), assuming a discontinuous initial N-wave, found some solutions valid for moderate times; these hold if the following conditions are met:

(i) the shock width must be small compared with the overall scale of the N-wave;

(ii) the correction due to diffusivity must not displace the shock too far from its location according to weak shock theory;

(iii) the Taylor shock solution itself remains valid as a leading order approximation.

They diagnosed the regions of space-time in which one or more of these conditions is violated and found new solutions of an asymptotic kind in each of these domains. For example in the case of spherical N-waves the complete solution is found with the exception of one region of space-time in which an irreducible nonlinear problem remains unsolved. In this region the outer limiting behaviour is, nevertheless, determined so that the solutions in all other regions are completely fixed. Even the form of the old age constant, actually a complicated function of two parameters, the initial Reynolds number and the spreading parameter, is completely determined by their analysis. For cylindrical N-waves an irreducible problem again results, but the motion can be followed right through into its old age phase, aside from an undetermined purely numerical constant. Correct results were obtained for the 'correction due to diffusivity' to the weak-shock theory prediction of shock centre location for plane, cylindrical and spherical N-waves.

We believe ours is the first exact representation of the solution for an important and physical generalized Burgers equation, arising from N-wave type of initial conditions. We hope to apply the method reported in the present paper to other generalized Burgers equations (Sachdev & Joseph 1994).

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