Limit laws for area coverage in non Boolean models

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Abstract. Sensor nodes being deployed randomly, one typically models its location by a point process in an appropriate space. The sensing region of each sensor is described by a sequence of i.i.d random sets. Hence sensor network coverage is generally analyzed by an equivalent coverage process. Properties of both area coverage and path coverage are well known in the literature for homogeneous sensor deployments. We study two models where the sensor nodes are deployed according to a stationary Cox point process and a non homogeneous Poisson point process respectively. We derive asymptotic properties of vacancy in both the models.

Key words. Cox point process, non homogeneous Poisson point process, Boolean model, area coverage, sensor networks.

AMS subject classifications (2000). Primary 60D05; 60G70
Secondary 60K37; 60G55

1 Introduction

1.1 Motivation

Coverage by a sensor network has always been a challenging problem from both theoretical and application viewpoint. A sensor is a device that measures a physical quantity over a region and converts it into a signal which can be read by an instrument or an observer. The union of all such sensing regions in the sensor field is the coverage provided by the sensor

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network. Coverage of a sensor network provides a measure of the quality of surveillance that the network can provide. Some of the common applications of sensor network includes environmental monitoring, emergency rescue, ambient control and surveillance networks. Adopting homogeneous scenarios in modeling the sensor locations is often too simplistic. For example, in applications like battlefield surveillance, an exact area of deployment is not known, at the same time a finite number of sensors are to be deployed over a large area. Even after deploying sensors, its location may change over time due to environmental factors like wind, river stream, rain, etc. Sometimes, a priori knowledge of the sensor field can be used to determine the concentration of sensors in the sensor field. This may result in higher concentration of sensors in some parts and lower concentration in others. Non uniform operational characteristics like interference, frequency of data collection and communication, etc., also results in non uniform degradation of the network. An appropriate way to model such deployments is to assume a non homogeneous distribution for the location of sensors. For instance, the stochastic environmental heterogeneity for the distribution of the sensor nodes may be modeled by a Cox point process. This motivates us to consider two such models for sensor deployment and study their coverage properties. In the first model we assume that the sensor locations are distributed according to a *stationary Cox point process* and in the later model we consider a *non homogeneous Poisson* distribution for the sensor locations.

1.2 Model Description

Let $\mathcal{P} \equiv \{\xi_i, i \geqslant 1\}$ be a stochastic point process in $\mathbb{R}^d, d \geq 1$ and $\{S_1, S_2, ...\}$ be i.i.d random sets in $\mathbb{R}^d$, independent of $\mathcal{P}$. Then $\mathcal{C} \equiv \{\xi_i + S_i, i \geqslant 1\}$ is called a coverage process. If in addition, $\mathcal{P}$ is a stationary Poisson point process then $\mathcal{C}$ is a Boolean model. The points of $\mathcal{P}$ may be interpreted as the location of sensors in a random sensor network and the shapes S_i may be thought of as the sensing area about the ith sensor. Instead of working with random sets S_i, we assume the sets $S_i' \ s$ to be a fixed (non random) non empty, Borel measurable subset (say S) of $\mathbb{R}^d$ with finite content c (i.e, $0 < c = \|S\| < \infty$). We consider the following two modes of deployment of sensors:-

(i) **Model I**

$\mathcal{P} \equiv \{\xi_i, \ i \geq 1\}$ be a Cox point process in $\mathbb{R}^d$. We have the following definition of Cox point process from [5] (pp. 186-187). Let $\wedge(x), \ x \in \mathbb{R}^d$, be a non negative random field (i.e, a non negative stochastic process indexed in $\mathbb{R}^d$) defined over some probability space $(\Omega, \mathcal{F}, P)$. Conditional on $\wedge(x) = \lambda(x)$, for $x \in \mathbb{R}^d$, let $\mathcal{P}(\lambda)$ be a non homogeneous Poisson process with intensity function λ. Then $\mathcal{P} \equiv \mathcal{P}(\wedge)$ is a *Cox point process.* We assume that $\wedge$ is stationary, i.e, $E[\wedge(x)] = \lambda$, a constant not depending on x.

(ii) **Model II**

$\mathcal{P} \equiv \{\xi_i,\ i \geq 1\}$ be a *non homogeneous Poisson process* in $\mathbb{R}^d$ with intensity function $\lambda(x)$. We assume the following bounds on the intensity function $\lambda(x)$, for every fixed $R \subseteq \mathbb{R}^d$, with finite content

$$0 \leq \lambda_l(R \pm S) \leq \lambda(x) \leq \lambda_u(R \pm S) < \infty \quad \forall\, x \in R \pm S, \tag{1.1}$$

the addition and substraction being understood in the Minkowski sense. (1.1) may be interpreted as the constraints on the number of sensors to be deployed in an operational area R, which we assume to be known for each fixed R. However in Section 3 we study the limit laws keeping R fixed. Hence we drop the argument and denote $\lambda_l(R \pm S)$ (respectively $\lambda_u(R \pm S)$) as λ_l (respectively λ_u) for the rest of the paper.

1.3 Previous Work

The coverage problem is one of the oldest in geometric probability. Coverage processes arising naturally in stochastic geometry have been studied in [12] along with their applications. However the Boolean model seems to be the most famous random set model in stochastic geometry. The area coverage properties for the Boolean model have been extensively studied in the literature, most notably in [5]. One may also see [8] for some recent results in the case of one coverage. The coverage of a line by a two dimensional Boolean model was investigated in [13]. In [10] the authors considers a coverage process where the shapes S_i have distributions that depend on the locations of their centers. They analyze the probability of the event that the whole of $\mathbb{R}^d$ will be covered by such a coverage process. The growth of tumor cells have been modeled using coverage process in [2]. The statistical properties of the coverage of a one dimensional path induced by a two dimensional non homogeneous random sensor network have been studied in [9]. In [7] coverage by a finite number of heterogeneous sensors is analyzed using integral geometry. The Cox process have been actively applied in Finance and Risk Theory [1, 6] and in forestry [11]. To the best of our knowledge, the problem of area coverage in the two models, described above has not been addressed before.

1.4 Organization of the Paper and Summary of Results

Our paper is in the same sprit as those that study area coverage (as in [5]). In Section 2 we consider a coverage process arising out of a Cox point process (Model I). We derive the expectation and variance of vacancy. Finally we study the asymptotic vacancy under suitable scaling and obtain a central limit theorem for the vacancy.

We carry out similar type of analysis in Section 3 when $\mathcal{P}$ is a non homogeneous Poisson process (Model II). We study the asymptotic vacancy under suitable conditions on the intensity function of the non homogeneous Poisson process and derive a central limit theorem for the vacancy. The techniques of the proofs in both cases are in general similar to those in Chapter 3 of [5]. We provide a detailed proof for the Cox process and indicate a sketch in the other case.

2 Coverage in a Cox point process (Model I)

2.1 Expectation and Variance of Vacancy

Consider a coverage process $\mathcal{C} \equiv \{\xi_i + S_i, i \geqslant 1\}$ when $\mathcal{P}$ is as in **(i)** of Section 1.2. Let R be a Borel subset of $\mathbb{R}^d, 0 < \|R\| < \infty$ and $S_i \equiv S \ \ \forall\, i \geq 1,$ S being a fixed (non random) non empty, Borel measurable subset of $\mathbb{R}^d$ with finite content c. Define the following indicator function for a point $x \in \mathbb{R}^d$

$$\chi(x) = \begin{cases} 1 & \text{if } x \notin \xi_i + S, \forall\ i \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define the vacancy V_C within R, to be the d-dimensional volume of the part not covered by $\mathcal{C}$, i.e.,

$$V_C = V_C(R) = \int_R \chi(x)\ dx. \tag{2.1}$$

By Fubini's theorem and stationarity of the Cox point process we obtain from [5]

$$\begin{aligned} E(V_C) &= E\left[\int_R \chi(x)dx\right] \\ &= \int_R Pr[\ x \notin \xi_i + S, \forall\ i \geq 1]\ dx \\ &= \int_R Pr[\ \xi_i \notin x - S, \forall\ i \geq 1]\ dx \\ &= \int_R Pr[\ \xi_i \notin -S, \forall\ i \geq 1]\ dx \\ &= \|R\| E\left[e^{-\int_{-S} \wedge(x)\ dx}\right]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} E[\chi(x_1)\ \chi(x_2)] &= E\left[e^{-\int_{(x_1-x_2-S)\cup(-S)} \wedge(x)\ dx}\right] \\ &= E\left[e^{-\left(\int_{x_1-x_2-S} \wedge(x)\ dx + \int_{-S} \wedge(x)\ dx - \int_{(x_1-x_2-S)\cap(-S)} \wedge(x)\ dx\right)}\right] \\ &= E\left[e^{-2\int_{-S} \wedge(x)\ dx}.e^{\int_{(x_1-x_2-S)\cap(-S)} \wedge(x)\ dx}\right]. \end{aligned} \tag{2.2}$$

The last line following from the stationarity of the Cox point process.

$$\begin{aligned} Cov[\chi(x_1)\ \chi(x_2)] &= E[\chi(x_1)\ \chi(x_2)] - E[\chi(x_1)]E[\chi(x_2)] \\ &= E\left[e^{-2\int_{-S} \wedge(x)\ dx}.e^{\int_{(x_1-x_2-S)\cap(-S)} \wedge(x)\ dx}\right] - \left(E\left[e^{-\int_{-S} \wedge(x)\ dx}\right]\right)^2. \end{aligned} \tag{2.3}$$

Hence

$$\begin{aligned} VAR(V_C) &= \int_R \int_R Cov[\chi(x_1)\ \chi(x_2)]\ dx_1\ dx_2 \\ &= \int_R \int_R \left(E\left[e^{-2\int_{-S} \wedge(x)\ dx}.e^{\int_{(x_1-x_2-S)\cap(-S)} \wedge(x)\ dx}\right] - \left(E\left[e^{-\int_{-S} \wedge(x)\ dx}\right]\right)^2\right)\ dx_1\ dx_2. \end{aligned} \tag{2.4}$$

2.2 Limit Laws for Model I

Consider a coverage process $\mathcal{C}(\delta)$ with fixed shapes S and $\mathcal{P}$ same as in **(i)** of Section 1.2. We scale the shapes S by δ ($\delta < 1$) in the $\mathcal{C}(\delta)$ model. Let V_C be the vacancy within the region R ($0 < \|R\| < \infty$) arising from the $\mathcal{C}(\delta)$ model. An excellent discussion for studying limit laws in scaled models can be found in Section 3.4 of [5]. In the analysis that follows, all stochastic convergence are in almost sure (a.s.) sense with respect to the probability measure P (refer **(i)** of Section 1.2).

Theorem 2.1. *Consider the scaled coverage process* $\mathcal{C}(\delta)$. *Let* $\delta \to 0$ *as* $\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \to \infty$ *a.s. such that* $\delta^d \left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) \to \rho$ *a.s.* $(0 < \rho < \infty)$, $\left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) VAR \left(e^{-\delta^d \int_{-S} \wedge(x)\ dx} \right) \to 0$ *a.s.* $\forall\ \mathcal{B} \subseteq \mathbb{R}^d,\ 0 < \|\mathcal{B}\| < \infty,\ \int_{\mathcal{B}} \wedge(x) > 0$ *a.s. and* $\delta^d \lambda \to l$, l *being any positive constant. Then,*

(i)

$$E(V_C) \to \| R \| \ e^{-\rho \|S\|}. \tag{2.5}$$

(ii)

$$VAR(V_C) \to 0. \tag{2.6}$$

(iii)

$$E|V_C - E(V_C)|^p \to 0, \qquad 1 \leq p < \infty. \tag{2.7}$$

(iv)

$$\left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) VAR(V_C) \xrightarrow{a.s.} \sigma^2 \equiv \rho \|R\| e^{-2\rho\|S\|} \int_{\mathbb{R}^d} \left[e^{\rho \|(y-S)\cap(-S)\|} - 1 \right] \ dy. \tag{2.8}$$

Theorem 2.2. *Consider the scaled coverage process* $\mathcal{C}(\delta)$. *Let* $\delta \to 0$ *as* $\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \to \infty$ *a.s. such that* $\delta^d \left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) \to \rho$ *a.s.* $(0 < \rho < \infty)$, $\left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) VAR \left(e^{-\delta^d \int_{-S} \wedge(x)\ dx} \right) \to 0$ *a.s.* $\forall\ \mathcal{B} \subseteq \mathbb{R}^d,\ 0 < \|\mathcal{B}\| < \infty,\ \int_{\mathcal{B}} \wedge(x) > 0$ *a.s. and* $\delta^d \lambda \to l$. *Then,*

$$\left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right)^{1/2} \{V_C - E(V_C)\} \xrightarrow{d} N(0, \sigma^2) \ \ a.s. \tag{2.9}$$

where σ^2 *is same as in (2.8).*

Remark 2.1. Recall $\lambda = E[\wedge(x)]$. From the scaling laws we have $\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \to \infty \ \ a.s. \Rightarrow \lambda \to \infty$. Hence the condition $\delta^d \lambda \to l$ is needed for the above results to hold.

Remark 2.2. If $\wedge(x) \equiv \gamma$, a constant then $VAR\left(e^{-\delta^d \int_{-S} \wedge(x)\ dx}\right) = 0$. Therefore the condition $\left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|}\right)\ VAR\left(e^{-\delta^d \int_{-S} \wedge(x)\ dx}\right) \to 0 \ \ a.s.$ is not required in [5] to prove the central limit theorem for vacancy in a Boolean model.

Remark 2.3. The limit laws of vacancy for the Boolean model can be derived from the above theorems by choosing $\wedge(x) \equiv \gamma$, a constant.

3 Coverage in a Non Homogeneous Poisson Process (Model II)

3.1 Expectation and Variance of Vacancy

Consider a coverage process $\mathcal{C} \equiv \{\xi_i + S_i, i \geqslant 1\}$ when $\mathcal{P}$ is as in **(ii)** of Section 1.2. Let R be a non empty Borel subset of $\mathbb{R}^d$ with finite content and $S_i \equiv S \ \ \forall\ i \geq 1,\ S$ being a fixed (non random) non empty, Borel measurable subset of $\mathbb{R}^d$ with finite content c. Define the following indicator function for a point $x \in \mathbb{R}^d$

$$\chi(x) = \begin{cases} 1 & \text{if } x \notin \xi_i + S, \forall\ i \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define the vacancy V_N within R, to be the d-dimensional volume of the part not covered by $\mathcal{C}$, i.e.,

$$V_N = V_N(R) = \int_R \chi(x)\ dx. \tag{3.1}$$

By similar calculations as in Section 2.1 we obtain

$$\begin{aligned} E(V_N) &= E\left[\int_R \chi(x) dx\right] \\ &= \int_R Pr[\ x \notin \xi_i + S, \forall\ i \geq 1]\ dx \\ &= \int_R Pr[\ \xi_i \notin x - S, \forall\ i \geq 1]\ dx \\ &= \int_R \left(e^{-\int_{x-S} \lambda(t)\ dt}\right)\ dx. \end{aligned} \tag{3.2}$$

Similarly, we have

$$
\begin{aligned}
E[\chi(x_1)\chi(x_2)] &= Pr[\ x_1 \notin \xi_i + S, \forall\ i \geq 1 \text{ and } \ x_2 \notin \xi_i + S, \forall\ i \geq 1\ ] \\
&= Pr[\ \xi_i \notin x_1 - S, \forall\ i \geq 1 \text{ and } \ \xi_i \notin x_2 - S, \forall\ i \geq 1\ ] \\
&= Pr[\ \xi_i \notin (x_1 - S) \cup (x_2 - S), \forall\ i \geq 1\ ] \\
&= e^{-\int_{(x_1-S)\cup(x_2-S)} \lambda(t)\ dt} \\
&= e^{-\left(\int_{x_1-S} \lambda(t)\ dt + \int_{x_2-S} \lambda(t)\ dt - \int_{(x_1-S)\cap(x_2-S)} \lambda(t)\ dt\right)}. \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
Cov[\chi(x_1)\ \chi(x_2)] &= E[\chi(x_1)\ \chi(x_2)] - E[\chi(x_1)]E[\chi(x_2)] \\
&= e^{-\left(\int_{x_1-S} \lambda(t)\ dt + \int_{x_2-S} \lambda(t)\ dt - \int_{(x_1-S)\cap(x_2-S)} \lambda(t)\ dt\right)} \\
&\quad - \left(e^{-\int_{x_1-S} \lambda(t)\ dt}\right)\left(e^{-\int_{x_2-S} \lambda(t)\ dt}\right). \qquad (3.4)
\end{aligned}
$$

Hence

$$
VAR(V_N) = \int_R \int_R e^{-\left(\int_{x_1-S} \lambda(t)\ dt + \int_{x_2-S} \lambda(t)\ dt\right)} \left[e^{\int_{(x_1-S)\cap(x_2-S)} \lambda(t)\ dt} - 1\right]\ dx_1\ dx_2. \qquad (3.5)
$$

We have the following bounds for $VAR(V_N)$ from (1.1)

$$
\begin{aligned}
& e^{-2\lambda_u \|S\|} \times \int_R \int_R \left[e^{\lambda_l \|(x_1 - x_2 + S)\cap S\|} - 1\right]\ dx_1\ dx_2 \\
& \leq VAR(V_N) \\
& \leq e^{-2\lambda_l \|S\|} \times \int_R \int_R \left[e^{\lambda_u \|(x_1 - x_2 + S)\cap S\|} - 1\right]\ dx_1\ dx_2. \qquad (3.6)
\end{aligned}
$$

3.2 Limit Laws for Model II

Consider a coverage process $\mathcal{C}(\delta)$ with fixed shapes S, $\mathcal{P}$ same as in **(ii)** of Section 1.2 and the intensity function $\lambda(x)$ of the non homogeneous Poisson point process satisfying the bound in (1.1). We scale the shapes S by δ ($\delta < 1$) in this $\mathcal{C}(\delta)$ model. Let V_N be the vacancy within the region R ($0 < \|R\| < \infty$) arising from the $\mathcal{C}(\delta)$ model. To obtain non trivial coverage we assume $\lambda_l > 0$ and $\lambda_u > 0$. We obtain the following results by same techniques as in [5].

Lemma 3.1. *If $\delta \to 0$, as $\lambda_l \to \infty$ and $\lambda_u \to \infty$, such that $\delta^d \lambda_l \to \rho_l$ and $\delta^d \lambda_u \to \rho_u$, where*

$0 < \rho_l \le \rho_u < \infty$, *then for the scaled process* $\mathcal{C}(\delta)$,

$$\limsup_{\lambda_l,\lambda_u\to\infty} E(V_N) \le \|R\| e^{-\rho_l\|S\|} \tag{3.7}$$

and

$$\liminf_{\lambda_l,\lambda_u\to\infty} E(V_N) \ge \|R\| e^{-\rho_u\|S\|}. \tag{3.8}$$

If $\rho_l = \rho = \rho_u$, *then*

$$E(V_N) \to \|R\| e^{-\rho\|S\|}. \tag{3.9}$$

Lemma 3.2. *If* $\delta \to 0$, *as* $\lambda_l \to \infty$ *and* $\lambda_u \to \infty$, *such that* $\delta^d \lambda_l \to \rho_l$ *and* $\delta^d \lambda_u \to \rho_u$, *where* $0 < \rho_l \le \rho_u < \infty$, *then for the scaled process,*

(i)

$$VAR(V_N) \to 0, \qquad (\textit{even if } \rho_l \ne \rho u)\,. \tag{3.10}$$

(ii)

$$E|V_N - E(V_N)|^p \to 0, \qquad 1 \le p < \infty. \tag{3.11}$$

(iii)

$$\limsup_{\lambda_l,\lambda_u\to\infty} \lambda_l VAR(V_N) \le \rho_l \|R\| e^{-2\rho_l\|S\|} \int_{\mathbb{R}^d} \left(e^{\rho_u\|(y+S)\cap S\|} - 1\right)\, dy \tag{3.12}$$

and

$$\limsup_{\lambda_l,\lambda_u\to\infty} \lambda_u VAR(V_N) \le \rho_u \|R\| e^{-2\rho_l\|S\|} \int_{\mathbb{R}^d} \left(e^{\rho_u\|(y+S)\cap S\|} - 1\right)\, dy. \tag{3.13}$$

(iv)

$$\liminf_{\lambda_l,\lambda_u\to\infty} \lambda_l VAR(V_N) \ge \rho_l \|R\| e^{-2\rho_u\|S\|} \int_{\mathbb{R}^d} \left(e^{\rho_l\|(y+S)\cap S\|} - 1\right)\, dy \tag{3.14}$$

and

$$\liminf_{\lambda_l,\lambda_u\to\infty} \lambda_u VAR(V_N) \ge \rho_u \|R\| e^{-2\rho_u\|S\|} \int_{\mathbb{R}^d} \left(e^{\rho_l\|(y+S)\cap S\|} - 1\right)\, dy. \tag{3.15}$$

Lemma 3.3. *If* $\rho_l = \rho = \rho_u$ *then under the* ***same scaling law*** *as in Lemma 3.1 we have for the scaled process,*

$$\begin{aligned} \lambda_i VAR(V_N) \to\ & \sigma^2 \\ & \equiv \rho \|R\| e^{-2\rho\|S\|} \int_{\mathbb{R}^d} \left(e^{\rho\|(y+S)\cap S\|} - 1\right)\, dy \quad \textit{for} \ \ i \in \{l,u\}. \end{aligned} \tag{3.16}$$

Lemma 3.4. *Consider the scaled coverage process $\mathcal{C}(\delta)$. If $\delta \to 0$, as $\lambda_l \to \infty$, and $\lambda_u \to \infty$, such that $\delta^d\lambda_l \to \rho$ and $\delta^d\lambda_u \to \rho$, $0 < \rho < \infty$, then for the scaled process,*

$$\lambda_i^{1/2}\{V_N - E(V_N)\} \xrightarrow{d} N(0,\sigma^2) \quad for \quad i \in \{l,u\} \tag{3.17}$$

where σ^2 is same as in (3.16).

Remark 3.1. As mentioned in Section 1.2 both ρ_u and ρ_l will depend on R and S. However, since R and S are kept fixed in the entire analysis, we do not indicate their dependence in the notation.

Remark 3.2. As $\lambda_l \to \infty$ and $\lambda_u \to \infty$, the intensity function $\lambda(x) \to \infty$, pointwise for all x in the area of interest. Hence the scaling law implies $\delta^d\lambda(x) \to \rho$, pointwise for all x in that region.

4 Proof of Main results

4.1 Proof of Results in Model I

We recall from Section 2.2, all almost sure convergence are with respect to the probability measure P.

Proof of Theorem 2.1. (2.5) follows trivially by dominated convergence theorem. By (2.4) and the inequality $e^x - 1 \le xe^x$, $x \ge 0$, we have,

$$\begin{aligned}
&VAR(V_C)\\
&= \int_R\int_R \left(E\left[e^{-2\int_{-\delta S}\wedge(x/\delta)\ dx} . e^{\int_{(x_1-x_2-\delta S)\cap(-\delta S)}\wedge(x/\delta)\ dx}\right] - \left(E\left[e^{-\int_{-\delta S}\wedge(x/\delta)\ dx}\right]\right)^2\right)\ dx_1\ dx_2\\
&\le \int_R\int_R E\left[\left(e^{-2\int_{-\delta S}\wedge(x/\delta)\ dx}\right)\left(\int_{(x_1-x_2-\delta S)\cap(-\delta S)}\wedge(x/\delta)\ dx\right)\left(e^{\int_{(x_1-x_2-\delta S)\cap(-\delta S)}\wedge(x/\delta)\ dx}\right)\right]\ dx_1\ dx_2\\
&\qquad\qquad + \|R\|^2 VAR\left(e^{-\int_{-\delta S}\wedge(x/\delta)\ dx}\right)\\
&\le \int_R\int_R E\left[\int_{(x_1-x_2-\delta S)\cap(-\delta S)}\wedge(x/\delta)\ dx\right]\ dx_1\ dx_2 \ + \ \|R\|^2 VAR\left(e^{-\int_{-\delta S}\wedge(x/\delta)\ dx}\right)\\
&\le \|R\|^2\left[\delta^d\lambda\|(\frac{x_1-x_2}{\delta} - S)\cap(-S)\| \ + \ VAR\left(e^{-\int_{-\delta S}\wedge(x/\delta)\ dx}\right)\right].
\end{aligned} \tag{4.1}$$

The last line following from Fubini's theorem and the fact that $E[\wedge(x)] = \lambda$. Both the terms in (4.1) converges to zero under the given scaling law and the fact that S has a finite content.

Hence (2.6) follows. By Chebychev's inequality,

$$P[|V_C - E(V_C)| > \epsilon] \leq \frac{VAR(V_C)}{\epsilon^2} \rightarrow 0.$$

Hence $V_C - E(V_C) \rightarrow 0$ in probability. Since $0 \leq V_C \leq \|R\|$, the dominated convergence theorem gives us the L^p convergence in (2.7). Applying the change of variables $x_1 - x_2 = y$ and $x_2 = x$, we obtain from (2.4)

$$VAR(V_C) = \delta^d \int_R dx \int_{\frac{(x-R)}{\delta}} \left(E\left[e^{-2\delta^d \int_{-S} \wedge(x)\ dx} . e^{\delta^d \int_{(y-S)\cap(-S)} \wedge(x)\ dx} \right] - \left(E\left[e^{-\delta^d \int_{-S} \wedge(x)\ dx} \right] \right)^2 \right) dy. \tag{4.2}$$

It is easy to see under the given scaling law

$$\begin{aligned} f_\delta(x) = \int_{\delta^{-1}(x-R)} & \left(E\left[e^{-2\delta^d \int_{-S} \wedge(x)\ dx} . e^{\delta^d \int_{(y-S)\cap(-S)} \wedge(x)\ dx} \right] - \left(E\left[e^{-\delta^d \int_{-S} \wedge(x)\ dx} \right] \right)^2 \right) dy \\ & \longrightarrow m \equiv e^{-2\rho\|S\|} \int_{\mathbb{R}^d} \left(e^{\rho\|(y-S)\cap(-S)\|} - 1 \right) dy. \end{aligned} \tag{4.3}$$

By dominated convergence theorem,

$$\left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) VAR(V_C) \rightarrow \rho \int_R m\ dx \equiv \rho m \|R\|,$$

as $\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \rightarrow \infty$ $a.s.$ Hence we have (2.8). □

Proof of Theorem 2.2. Let r be a large positive constant. We divide all of $\mathbb{R}^d$ into a regular lattice of d dimensional cubes of side length $(\tau r \delta)$,where $\tau = 2c = 2\|S\|$, so that each cube is separated from its adjacent cube by a spacing strip of width $(2\tau\delta)$. Let A_1 denote the union of all the cubes which are wholly contained within R. A_2 be the union of all the spacing strips that are wholly contained within R. Finally we denote A_3 as the intersection with R of all the spacing strips and the cubes that are contained only partially within R. The above configuration is illustrated in Figure 1 for dimension d=2. Let $V_C^{(i)}$ be the vacancy within the region A_i. The vacancy within R can be expressed as,

$$V = V_C^{(1)} + V_C^{(2)} + V_C^{(3)}.$$

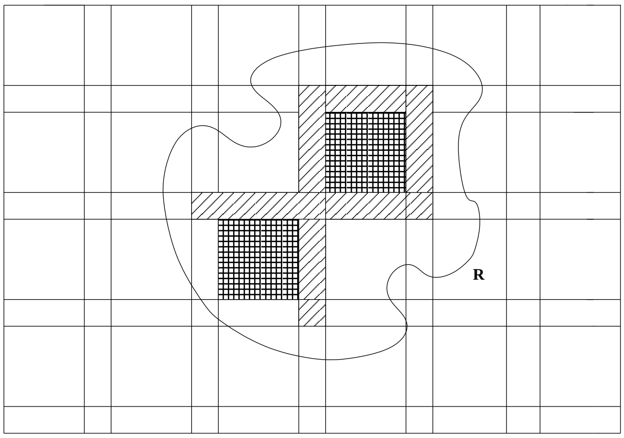

Figure 1: The region shaded in dark represents A_1 whereas the lightly shaded region represents A_2. The unshaded part within the region R represents A_3.

As $\delta \to 0$, under the assumption of the theorem the cubes gets finer and we have

$$\|A_3\| \to 0 \ \ a.s. \tag{4.4}$$

We also have,

$$\|A_2\| \leq \frac{l}{r}, \qquad \text{where } l \text{ is a constant independent of r.} \tag{4.5}$$

The vacancy $V_C^{(i)}$ within the region A_i satisfies

$$\begin{aligned}
&VAR(V_C^{(i)}) \\
&= \int_{A_i}\int_{A_i} \left(E\left[e^{-2\int_{-\delta S} \wedge(x/\delta)\ dx} . e^{\int_{(x_1-x_2-\delta S)\cap(-\delta S)} \wedge(x/\delta)\ dx} \right] - \left(E\left[e^{-\int_{-\delta S} \wedge(x/\delta)\ dx} \right] \right)^2 \right)\ dx_1\ dx_2 \\
&\leq \int_{A_i}\int_{A_i} E\left[\left(e^{-2\int_{-\delta S} \wedge(x/\delta)\ dx} \right) \left(\int_{(x_1-x_2-\delta S)\cap(-\delta S)} \wedge(x/\delta)\ dx \right) \left(e^{\int_{(x_1-x_2-\delta S)\cap(-\delta S)} \wedge(x/\delta)\ dx} \right) \right]\ dx_1\ dx_2 \\
&\qquad\qquad\qquad\qquad + \ \|A_i\|^2 VAR\left(e^{-\int_{-\delta S} \wedge(x/\delta)\ dx} \right) \\
&\leq \int_{A_i}\int_{\mathbb{R}^d} E\left[\int_{(x_1-x_2-\delta S)\cap(-\delta S)} \wedge(x/\delta)\ dx \right]\ dx_1\ dx_2 \ + \ \|A_i\|^2 VAR\left(e^{-\int_{-\delta S} \wedge(x/\delta)\ dx} \right) \\
&\leq \|A_i\| \delta^{2d} \lambda(\|S\|)^2 \ + \ \|A_i\|^2 VAR\left(e^{-\int_{-\delta S} \wedge(x/\delta)\ dx} \right).
\end{aligned} \tag{4.6}$$

The last line following from Fubini's theorem and stationarity of the Cox process. From

(4.4), (4.5) and Remark 2.1, we have,

$$\lim_{\int_{\mathcal{B}} \wedge(x)\ dx \to \infty} \left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) VAR(V_C^{(3)}) = 0 \ \ a.s. \tag{4.7}$$

and

$$\lim_{r\to\infty} \limsup_{\int_{\mathcal{B}} \wedge(x)\ dx \to \infty} \left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \right) VAR(V_C^{(2)}) = 0 \ \ a.s. \tag{4.8}$$

Hence we observe that the only significant term involves $V_C^{(1)}$ and to prove the central limit theorem it is enough to show that,

$$\{V_C^{(1)} - E(V_C^{(1)})\}/(VAR(V_C^{(1)}))^{1/2} \xrightarrow{d} N(0,1) \ \ a.s. \tag{4.9}$$

and

$$\lim_{r\to\infty} \limsup_{\int_{\mathcal{B}} \wedge(x)\ dx \to \infty} \left| \frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|} \left(VAR(V_C^1) - VAR(V_C) \right) \right| = 0 \ \ a.s. \tag{4.10}$$

Let n denote the number of small cubes of length $\tau r \delta$ which make up the region A_1, and let D_i denote the ith of these cubes, for $1 \le i \le n$. We further denote by U_i the contribution to $V_C^{(1)}$ from D_i. Then we have $V_C^{(1)} = \sum_{i=1}^n U_i$. Since the shape δS is contained within a sphere of radius $\tau\delta$, and the cubes D_i are least $2\tau\delta$ distance apart, the shape δS can not intersect more than one cube. Due to stationarity of the Cox point process the variables U_i are identically distributed. Hence U_i' s are i.i.d random variables and we have

$$VAR(V_C^{(1)}) = \sum_i^n VAR(U_i) = nVAR(U_i).$$

Let D be a d dimensional cube of side length τr having the same orientation as D_1. For any two real sequences a_n and b_n $a_n \sim b_n$ implies $a_n/b_n \to 1$ as $n \to \infty$. From (2.4) we obtain

$$\begin{aligned} & VAR(V_C^{(1)}) \\ & = n \int_{D_i} \int_{D_i} \left(E\left[e^{-2\int_{-\delta S} \wedge(x/\delta)\ dx} . e^{\int_{(x_1 - x_2 - \delta S) \cap (-\delta S)} \wedge(x/\delta)\ dx} \right] - \left(E\left[e^{-\int_{\delta S} \wedge(x/\delta)\ dx} \right] \right)^2 \right) dx_1\ dx_2 \\ & \sim n\delta^{2d} e^{-2\rho\|S\|} \int_D \int_D \left(e^{\rho \|(x_1 - x_2 - S) \cap (-S)\|} - 1 \right) dx_1\ dx_2. \end{aligned}$$

Since $n = O\left(\int_{\mathcal{B}} \wedge(x)\ dx\right)$ *a.s.* we have

$$\begin{aligned}
\frac{\sum_i E|U_i - E(U_i)|^3}{(\sum_i Var(U_i))^{3/2}} &= \frac{E|U_i - E(U_i)|^2 E|U_i - E(U_i)|}{(\sum_i Var(U_i))^{3/2}} \\
&\leq \frac{2 \parallel D_i \parallel Var(U_i)}{(\sum_i Var(U_i))^{3/2}} \\
&= \frac{2(\tau r \delta)^d}{(\sum_i Var(U_i))^{1/2}} = O\left(\left(\int_{\mathcal{B}} \wedge(x)\ dx\right)^{-1/2}\right) \to 0 \qquad (4.11)
\end{aligned}$$

as $\int_{\mathcal{B}} \wedge(x)\ dx \to \infty$ *a.s.* Hence (4.9) follows by Lyapunov's central limit theorem. By Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left|VAR(V_C) - VAR(V_C^{(1)})\right| \\
&= \left|E\left[V_C - E\left(V_C\right)\right]^2 - E\left[V_C - E\left(V_C^{(1)}\right)\right]^2\right| \\
&= \left|E\left[\left(V_C^{(2)} + V_C^{(3)} - E\left(V_C^{(2)} + V_C^{(3)}\right)\right) \times \left(2V_C - V_C^{(2)} - V_C^{(3)} - E\left(2V_C - V_C^{(2)} - V_C^{(3)}\right)\right)\right]\right| \\
&\leq 4\left[VAR\left(V_C^{(2)}\right) + VAR\left(V_C^{(3)}\right)\right]^{1/2}\left[VAR\left(V_C\right) + VAR\left(V_C^{(2)}\right) + VAR\left(V_C^{(3)}\right)\right]^{1/2}. \qquad (4.12)
\end{aligned}$$

(4.10) follows from (2.8), (4.7), (4.8) and (4.12). Hence as $\int_{\mathcal{B}} \wedge(x)\ dx \to \infty$ *a.s.* we have Theorem 2.2. □

Remark 4.1. The condition $\left(\frac{\int_{\mathcal{B}} \wedge(x)\ dx}{\|\mathcal{B}\|}\right)\ VAR\left(e^{-\delta^d \int_{-S} \wedge(x)\ dx}\right) \to 0$ *a.s.* is used only in showing (4.3), (4.7) and (4.8).

4.2 Proof of Results in Model II

By using asymptotic properties of vacancy for the Boolean model (as in [5], pp. 141-158), one can derive all the results for asymptotic vacancy in Section 3.2. While investigating the asymptotic properties of vacancy for the Cox point process in Section 2.2, we have modified the proofs given in [5] and illustrated the detailed techniques. Hence to avoid repetitions we indicate only a sketch of the proofs in the non homogeneous model. In all the proof that follows the operational area R is kept fixed.

Proof of Lemma 3.1. We have from (3.2)

$$\|R\|e^{-\lambda_u\|S\|} \leq E(V_N) \leq \|R\|e^{-\lambda_l\|S\|}. \tag{4.13}$$

Thus the expected vacancy is bounded from above (respectively from below) by the expectation of vacancy in a Boolean model driven by a Poisson point process with intensity λ_l (respectively λ_u) and generated by the same shapes S. Now scaling the shapes S by δS we have,

$$\|R\|e^{-\lambda_u\delta^d\|S\|} \leq E(V_N) \leq \|R\|e^{-\lambda_l\delta^d\|S\|}.$$

Hence (3.7) and (3.8) follows under the scaling law in Lemma 3.1. In the case when $\rho_u = \rho = \rho_l$, (3.9) follows trivially. □

Proof of Lemma 3.2. From equation (3.6) for the scaled process we have

$$VAR(V_N) \leq e^{-2\lambda_l\delta^d\|S\|} \times \int_R\int_R \left[e^{\left(\lambda_u\delta^d\|(\frac{x_1-x_2}{\delta}+S)\cap S\|\right)} - 1\right]\ dx_1\ dx_2. \tag{4.14}$$

(3.10) now follows by the same argument as in (2.6) of Theorem 2.1. By Chebychev's inequality,

$$P[|V_N - E(V_N)| > \epsilon] \leq \frac{VAR(V_N)}{\epsilon^2} \to 0.$$

Hence $V_N - E(V_N) \to 0$ in probability. Since $0 \leq V_N \leq \|R\|$, the dominated convergence theorem gives us the L^p convergence in (3.11). By making the change of variable, $x_1 - x_2 = y$ and $x_2 = x$, as in the proof of (2.8) in Theorem 2.1 one can prove (3.12), (3.13), (3.14) and (3.15). □

Lemma 3.3 follows trivially from Lemma 3.2 when $\rho_l = \rho = \rho_u$.

Proof of Lemma 3.4. We proceed exactly as in the proof of Theorem 2.2. Unless otherwise stated all the notations that are used in the proof of Theorem 2.2 will have an analogous meaning for Model II. We obtain the following equations for the scaled process from (3.6) and [5].

$$\begin{aligned} VAR(V_N^{(i)}) &\leq e^{-2\lambda_l\|\delta S\|} \times \int_{A_i}\int_{A_i} \left[e^{(\lambda_u\|(x_1-x_2+\delta S)\cap\delta S\|)} - 1\right]\ dx_1\ dx_2 \\ &\leq \lambda_u\delta^{2d}e^{\lambda_u\delta^d\|S\|}\|A_i\|\|S\|^2. \end{aligned} \tag{4.15}$$

Hence

$$\lim_{\lambda_u \to \infty} \lambda_u VAR\left(V_N^{(3)}\right) = 0 \tag{4.16}$$

and

$$\lim_{r \to \infty} \limsup_{\lambda_u \to \infty} \lambda_u VAR\left(V_N^{(2)}\right) = 0. \tag{4.17}$$

Hence to prove the central limit theorem it is sufficient to prove that

$$\{V_N^{(1)} - E(V_N^{(1)})\}/(VAR(V_N^{(1)})^{1/2} \xrightarrow{d} N(0,1), \tag{4.18}$$

and

$$\lim_{r \to \infty} \limsup_{\lambda_u \to \infty} \left|\lambda_u \left(VAR\left(V_N^{(1)}\right) - VAR(V_N)\right)\right| = 0. \tag{4.19}$$

$$\begin{aligned}
\frac{\sum_{i=1}^n E|U_i - E(U_i)|^3}{(\sum_{i=1}^n VAR(U_i))^{3/2}} &\leq \frac{(\tau r \delta)^d}{(\sum_{i=1}^n Var(U_i))^{1/2}} \\
&\leq \frac{(\tau r \delta)^d}{(ne^{-2\lambda_u \|\delta S\|} \times \int_{D_i}\int_{D_i} [e^{\lambda_l \|(x_1 - x_2 + \delta S) \cap \delta S\|} - 1] \; dx_1 \; dx_2)^{1/2}} \\
&= O(\lambda_u^{-1/2}) \longrightarrow 0 \quad \text{under the given scaling law.}
\end{aligned} \tag{4.20}$$

Therefore (4.18) follows by Lyapunov's central limit theorem. The rest of the proof follows as in ([5] pp. 157-158). The other statement in Lemma 3.4 can be proved in a similar way. □

5 Related Problems

The results of [5] for asymptotic vacancy in a Boolean model have been generalized in [4], using the notion of "associated random measures". It could be interesting to obtain similar generalization results for the coverage processes studied in our paper. It is not clear if one can obtain a central limit theorem for the vacancy in non homogeneous deployment of sensors without imposing condition (1.1) on the intensity function. One can carry out a similar type of analysis of path coverage as in [9, 13] for the Cox point process.

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