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On the Number of Minimum Cuts in a Graph

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Abstract. We relate the number of minimum cuts in a weighted undirected graph with various structural parameters of the graph. In particular, we upper–bound the number of minimum cuts in terms of the radius, diameter, minimum degree, maximum degree, chordality, expansion, girth etc. of the graph.

1 Introduction

Let $G = (V, E)$ be a graph or a multi-graph with positive weights on its edges. We denote $|V|$ by n and $|E|$ by m. By an unweighted graph, we mean that all the edges have unit weight. Let $(A, \overline{A})$ denote a cut of G, defined by the subsets $A \subset V$ and $\overline{A} = V - A$. We denote by $E(A, \overline{A})$, the set of edges in the cut, i.e., $E(A, \overline{A}) = \{(u, v) \in E : u \in A \textit{ and } v \in \overline{A}\}$. The weight of the cut $(A, \overline{A})$ is defined as the sum of weights on all the edges in $E(A, \overline{A})$, and will be denoted by $w(A, \overline{A})$. A minimum cut $(S, \overline{S})$ is one with the minimum weight over all cuts in G. (Some authors use the words *global minimum cuts* or *connectivity cuts* instead of minimum cuts). We will denote the weight of the minimum cut in G by $\lambda(G)$. Note that if G is unweighted, $\lambda(G)$ is same as the edge connectivity of the graph, i.e., the minimum number of edges whose removal disconnects the graph.

Note that the minimum cut in a graph may not be unique. We use $\Lambda(G)$ to denote the number of minimum cuts in G. The problem of counting the number of minimum cuts in a weighted undirected graph arises in various aspects of network reliability, like testing the super–λ–ness of a graph [2], estimating the probabilistic connectedness of a stochastic graph in which edges are subject to failure with probability p [4–6, 16] and other areas [17]. For example, for a sufficiently small p, the probabilistic connectedness of G can be approximated as $P(G, p) \approx 1 - \Lambda(G)p^{\lambda(G)}(1-p)^{|E|-\lambda(G)}$, suggesting the importance of counting and bounding $\Lambda(G)$.

It is well known that for any weighted graph G, $\Lambda(G) \le \binom{n}{2}$ and this upper bound is achieved if G is a cycle C_n of n nodes with each edge having weight $\frac{\lambda(G)}{2}$ [7, 1, 11]. It is interesting to explore whether there exist tighter bounds for

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$\Lambda(G)$ when the graph satisfies various properties. For example, Bixby [1] studies $\Lambda(G)$ in terms of the weight of the minimum cuts $\lambda(G)$, in the special case where all the edge weights are positive integers and $\lambda(G)$ is an odd integer. For this case, Bixby [1] shows that $\Lambda(G) \leq \lfloor \frac{3n}{2} \rfloor - 2$. In the case of unweighted simple graphs it is shown by Lehel, Maffray and Priessmann [12] that if $\lambda(G) = k$ where $k \geq 4$ is an even positive integer, then $\Lambda(G) \leq \frac{2n^2}{(k+1)^2} + \frac{(k-1)n}{k+1}$. When $k > 5$ is an odd integer, they show that $\Lambda(G) \leq (1 + \frac{4}{k+5})n$. The inherent structural difference between graphs with odd and even edge connectivity was pointed out by Kanevsky [13] also.

In this paper, we provide upper bounds for $\Lambda(G)$ in terms of many other important parameters of graphs. We assume weighted graphs. Multi-graphs, as far as the results here are concerned, can be considered as a special case of weighted graphs, since the multi-edges can be replaced by a single edge of appropriate weight without affecting the value of $\Lambda(G)$. Our only assumption about the weights is that they are positive. Note that, for the purposes of this paper, this assumption is equivalent to the assumption that the weights are at least 1, since multiplying the weights on every edge by the same constant will not change $\Lambda(G)$. While our upper bounds are valid for weighted undirected graphs and multi graphs, in most cases, the properties in terms of which the upper bounds are stated depend only on the structure of those graphs. In other words, the radius or minimum degree in terms of which we describe the upper bounds are those of the underlying unweighted simple graph and do not depend on the weights of the edges.

There is an abundance of literature regarding the determination of $\lambda(G)$ and finding a minimum cut in G. The problem of enumerating all the minimum cuts is considered by many authors [7, 15, 10, 8], and various structures (for example, the cactus structure) are invented to efficiently represent all the minimum cuts in a graph. The fact that the performance of some of these algorithms depends on the number of minimum cuts in the graph also makes it interesting to look for tighter upper bounds for $\Lambda(G)$ when G satisfies certain properties. (For example a randomized algorithm due to Karger builds a data structure that represents all minimum cuts in $O(\Lambda(G) + n \log n)$ space). See [10] for a brief survey of results regarding the enumeration of all minimum cuts.

The slightly different question of upper–bounding the number of approximate minimum cuts– i.e., those cuts having weight at most $f\lambda(G)$, where $f > 1$ is a constant, is considered in [18, 11, 14, 19] etc. For example, Karger [11], using probabilistic analysis shows that there are at most $O(n^{2f})$ cuts of the above kind in a graph of n nodes. Nagamochi et.al. [14] show that the number of cuts of weight at most $\frac{4}{3}\lambda(G)$ is upper–bounded by $\binom{n}{2}$. Williamson and Henzinger [19] show an upper bound of $O(n^2)$ for the number of cuts of weight at most $\frac{3}{2}\lambda(G)$, extending the arguments of [14].

1.1 Our results

Radius and Diameter : If $G = (V, E)$ is a connected graph, the *eccentricity* of a node $v \in V$ is defined as $e(v) = \max distance(v, u)$ over all the nodes $u \in V$.

The radius of the graph G, $r(G) = \min_{v \in V} e(v)$. A vertex v is a central node if $e(v) = r(G)$. Diameter of G, $d(G) = max_{v \in V} e(v)$. (Note that, in this paper, by "distance", we mean only the distances in the underlying unweighted graph. Thus radius, eccentricity, diameter etc. have nothing to do with the weights). We show that $\Lambda(G) \leq (r+1)n - (2r+1) \leq (d+1)n - (2d+1)$, where G is a weighted graph and r, d are the radius and diameter of G. As a special case, we observe that if there is a node which is a neighbour of every other node in the graph, i.e., if $r(G) = 1$, then $\Lambda(G) \leq 2n - 3$. We illustrate the tightness of this bound by constructing a weighted clique $\mathcal{K}_n$ for which $\Lambda(\mathcal{K}_n) = 2n - 3$.

Minimum and Maximum Degree : Let the minimum degree and maximum degree of G be δ and Δ respectively. (Note that minimum and maximum degrees have nothing to do with the weights, i.e., $\delta = \min_{u \in V} |N(u)|$ and $\Delta = \max_{u \in V} |N(u)|$, $N(u)$ being the set of neighbours of the node u). We show that $\Lambda(G) \leq (\frac{3n}{2(\delta+1)} + 1.5)n - (\frac{3n}{\delta+1} + 2)$ and $\Lambda(G) \leq \frac{(n-\Delta+3)n}{2} - (n - \Delta + 2)$. Note that these bounds become significant when the concerned parameters are reasonably large. Also it is easy to get an upper bound involving both δ and Δ, by extending the techniques discussed in the paper.

Chordality : Let C be a simple cycle of a weighted undirected graph G. Any edge in the induced subgraph on the nodes of C, $G(C)$, other than the cycle edges themselves is called a chord of C. C is called an induced cycle (or chordless cycle), iff C does not have any chords. The length of the largest induced cycle in a graph G is called chordality of G. A graph G is called k–chordal iff the chordality of G is at most k. $\Lambda(G) \leq \frac{(k+1)n}{2} - k$, where k is the chordality of the underlying unweighted (simple) graph, corresponding to G. We also show the tightness of the bound by exhibiting a k-chordal graph G for arbitrarily large n, such that $\Lambda(G) = \frac{(k+1)n}{2} - k$.

The word "chordality" originates from the well known subclass of perfect graphs, the chordal graphs. A graph G is chordal iff there is no induced cycle of length 4 or more in G. We define the chordality of a chordal graph to be 3. All graphs other than chordal graphs have chordality ≥ 4. Some other important classes of graphs with low chordality value are the cocomparability graphs, chordal bipartite graphs and weakly chordal graphs, all of which are known to be 4–chordal.

Note that C_n (the cycle on n nodes) is the graph with maximum chordality amongst all graphs on n nodes. Also, it is a graph which contains the maximum number of minimum cuts possible, namely $\binom{n}{2}$. (In fact our bound given above, shows that C_n with each edge having weight $\frac{\lambda}{2}$ is the *only* graph which contains $\binom{n}{2}$ minimum cuts, the weight of the minimum cut being λ). The fact that the maximum value of $\Lambda(G)$ is achieved by the graph of largest chordality motivates a study of the influence of chordality on $\Lambda(G)$.

Edge Expansion : Let $E(X, \overline{X})$ be the set of edges in the cut $(X, \overline{X})$. Then the edge expansion $\beta_e(G)$ is defined as $\beta_e(G) = \min \frac{|E(X,\overline{X})|}{|X|}$, over all subsets X of V with $|X| \leq \frac{n}{2}$. We define the weighted edge expansion, $\beta_w(G) = \min \frac{w(X,\overline{X})}{|X|}$ over all subsets X of V, such that $|X| \leq \frac{n}{2}$. A related concept is that of vertex expansion (or simply the expansion), $\beta_v(G)$, which is defined as $\beta_v(G) = min\frac{|N(X)|}{|X|}$ over all subsets X of V, such that $|X| \le \frac{n}{2}$ where $N(X)$ denotes the set of neighbours of X, namely the set $\{u \in V - X : (u,v) \in E \ for \ some \ v \in X\}$.

We show that $\Lambda(G) \le \frac{(\frac{\lambda}{\beta}+3)}{2}n - (\frac{\lambda}{\beta}+2)$ where $\beta = \beta_w(G)$. It is easy to note that (since we can assume without any loss of generality that all weights are at least 1) $\beta_w(G) \ge \beta_e(G) \ge \beta_v(G)$. Thus we can substitute $\beta_v(G)$ or $\beta_e(G)$ in the place of β in the above formula.

Expansion is a concept which has found application in areas ranging from complexity theory to coding theory. A further motivation for looking for a bound in terms of the expansion properties of the graph is the well known fact that *almost all graphs* are good expanders, i.e., if we randomly construct a graph by selecting each edge with certain probability p (greater than a threshold value), the probability that the expansion of the resulting graph being bounded below by a constant tends to 1 as $n \to \infty$.

Girth : Girth is the length of the smallest (induced) cycle in G. The intuition that expansion properties can control the number of minimum cuts in a graph, leads us to a bound in terms of girth also. We show that if G is an unweighted graph with girth g and minimum degree δ, the $\Lambda(G) < (\frac{n}{x+1}+1)n - (\frac{2n}{x+1}+1)$, where $x = \left(\frac{\delta-3}{2}\right)^{\frac{g-3}{2}}$. Note that this is in contrast with the bound in terms of chordality, the length of the largest induced cycle.

2 Preliminaries

Consider an undirected graph $G = (V,E)$ with a weight function $w : E \to \Re^+$. Let U and W be disjoint subsets of V. Let $E(U,W) = \{(u,v) \in E : u \in U, v \in W\}$, be the set of edges between the vertices in U and the vertices in W. Also, let $w(U,W)$ be the sum of the weights on the edges in $E(U,W)$. We denote the induced subgraph on the set of nodes U by $G(U)$. The set of neighbours of a node $u \in V$, will be denoted by $N(u)$, i.e., $N(u) = \{v \in V : (u,v) \in E\}$. Also $N(U) = \bigcup_{u \in U} N(u) - U$. As mentioned in the introduction, $\lambda(G)$ denotes the weight of a minimum cut and $\Lambda(G)$ denotes the number of minimum cuts in G.

Lemma 1. *If $(S,\overline{S})$ is a minimum cut of a connected undirected graph G, then $G(S)$ and $G(\overline{S})$ are connected.*

Proof. Suppose that $G(S)$ is not connected. Let $G(S_1)$ be a connected component of $G(S)$, where $S_1 \subset S$. Clearly $(S_1,\overline{S_1})$ is a cut of G and $w(S_1,\overline{S_1}) < w(S,\overline{S})$, since $E(S_1,\overline{S_1}) \subset E(S,\overline{S})$. But this is a contradiction since $(S,\overline{S})$ is assumed to be a minimum cut. ■

Definition 1. *Let $(X,\overline{X})$ and $(Y,\overline{Y})$ be two cuts in a weighted undirected graph. $(X,\overline{X})$ and $(Y,\overline{Y})$ are said to cross each other iff all the four sets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$ are non empty. Then $(X,\overline{X})$ and $(Y,\overline{Y})$ are called a crossing pair of cuts.*

Lemma 2. *A pair of cuts* $(S,\overline{S})$ *and* $(P,\overline{P})$ *does not cross if and only if* S(*or* $\overline{S}$) *is a subset of* P *or* $\overline{P}$. (*i.e.,* $S \subseteq P$, $S \subseteq \overline{P}$, $\overline{S} \subseteq P$ *or* $\overline{S} \subseteq \overline{P}$)

Proof. Follows from the definition of a crossing pair of cuts. ■

Lemma 3 (Bixby [1], Dinitz, Karzanov and Lomosonov [7]). *Let* $(X,\overline{X})$ *and* $(Y,\overline{Y})$ *be a crossing pair of minimum cuts in a weighted undirected graph* G. *Let* $A = X \cap Y$, $B = \overline{X} \cap Y$, $C = X \cap \overline{Y}$ *and* $D = \overline{X} \cap \overline{Y}$. *Then,*

1. $w(A,B) = w(B,D) = w(D,C) = w(C,A) = \frac{\lambda(G)}{2}$
2. $w(A,D) = w(B,C) = 0$. *That is,* $E(A,D) \cap E(B,C) = \emptyset$

Lemma 4. *If* $(P,\overline{P})$ *and* $(S,\overline{S})$ *are a crossing pair of minimum cuts, then* $E(P,\overline{P}) \cap E(S,\overline{S}) = \emptyset$.

Proof. $E(P,\overline{P}) \cap E(S,\overline{S}) = E(S \cap P, \overline{S} \cap \overline{P}) \cup E(\overline{S} \cap P, S \cap \overline{P}) = \emptyset$, by Lemma 3. ■

Definition 2. *A circular partition* $\mathcal{C} = (U_0, U_1, U_2, \cdots, U_{k-1})$ *of the vertices of a graph* G *is a partition of the set of vertices* V *of* G *into disjoint non–empty subsets* $U_0, U_1, \cdots, U_{k-1}$ *such that*

1. $w(U_i, U_{i+1 \ mod \ k}) = \frac{\lambda(G)}{2}$, *for* $0 \leq i \leq k-1$.
2. $w(U_i, U_j) = 0$, *if* $i \neq j+1 \ mod \ k$ *or* $i \neq j-1 \ mod \ k$, *i.e.,* $E(U_i, U_j) = \emptyset$.
3. *For* $0 \leq i \leq k-1$, *the cut* $(U_i, \overline{U}_i)$ – *which is a minimum cut by conditions* 1 *and* 2 – *does not cross with any other minimum cut* $(A, \overline{A})$ *in* G.

Definition 3. *A cut* $(A,\overline{A})$ *is called a union cut with respect to a circular partition* $\mathcal{C} = (U_0, U_1, \cdots, U_{k-1})$ *iff* $A = \bigcup_{j=i}^{i+b-1 \ mod \ k} U_j$ *where* $2 \leq b \leq k-2$. (*Note that both* A *and* $\overline{A}$ *contain at least 2 subsets in* $\mathcal{C}$).

The cut $(A,\overline{A})$ *is called a subset cut with respect to* $\mathcal{C}$ *iff* $A \subseteq U_i$ *or* $\overline{A} \subseteq U_i$ *for some* i.

Lemma 5. *Let* $\mathcal{C} = (U_0, U_1, \cdots, U_{k-1})$ *be a circular partition of* G. *Then any minimum cut* $(S,\overline{S})$ *of* G *is either a union cut or a subset cut with respect to* $\mathcal{C}$. *Moreover, every union cut with respect to* $\mathcal{C}$ *is a minimum cut in* G.

Proof. By the definition of a circular partition, $(S,\overline{S})$ does not cross with any of the minimum cuts $(U_i, \overline{U}_i)$. Therefore, by Lemma 2, $S \subseteq U_i$, $\overline{S} \subseteq U_i$, $S \subseteq \overline{U}_i$ or $\overline{S} \subseteq \overline{U}_i$. Suppose $(S,\overline{S})$ is not a subset cut. Then, we have $U_i \subset \overline{S}$ or $U_i \subseteq S$, for all i. Since by Lemma 1, $G(S)$ and $G(\overline{S})$ are connected, we infer that $(S,\overline{S})$ is a union cut. Also, that a union cut is a minimum cut follows easily from conditions 1 and 2 of the definition of a circular partition. ■

Lemma 6. *Let* G *be a weighted undirected graph. Then* G *has a circular partition* $\mathcal{C} = (U_0, U_1, \cdots, U_{k-1})$, *where* $k \geq 4$, *if and only if there exists a crossing pair of minimum cuts in* G.

Proof. If there is a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{k-1})$ with $k \geq 4$, clearly the minimum cuts $(U_0 \bigcup U_1, \overline{U_0 \bigcup U_1})$ and $(U_1 \bigcup U_2, \overline{U_1 \bigcup U_2})$ cross with each other. On the other hand, if there is a crossing pair of minimum cuts in G namely $(S_1, \overline{S}_1)$ and $(S_2, \overline{S}_2)$, due to a Theorem of Bixby [1], Dinits, Karzanov, Lomosonov [7], etc. there exists a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{k-1})$ such that each of $S_1 \cap S_2$, $\overline{S}_1 \cap S_2$, $S_1 \cap \overline{S}_2$ and $\overline{S}_1 \cap \overline{S}_2$ equal $\bigcup_{i=a}^{b-1} U_i$ for appropriate choices for a and b. The if part of the Lemma follows immediately from this. ■

For a circular partition $\mathcal{C}$ of G, let the *partition number* $p(\mathcal{C})$ be defined as the number of subsets in $\mathcal{C}$. We define the *partition number of the graph* G as follows.

Definition 4. *The partition number $p(G)$ of a graph G is defined as $p(G) = 3$, if there is no circular partition for G. Otherwise, $p(G) = \max p(\mathcal{C})$, over all circular partitions $\mathcal{C}$ of G.*

Note that if there is a crossing pair of minimum cuts in G, then $p(G) \geq 4$, by Lemma 6. Otherwise, $p(G) = 3$.

Definition 5. *By contraction of a subset of vertices $X \subset V$, we mean replacing all the vertices in X by a single vertex x and adding the edges (y, x) for each $y \in N(X)$. The weight of the edge (y, x) (where $y \in N(X)$) is assigned to be $w(y, x) = \sum_{z \in X} w(y, z)$, where $(y, z) \in E(G)$. We denote the graph obtained after the contraction operation by G/X. We will refer to the operation of undoing the effect of a contraction (i.e., restoring G from G/X) by putting back X in the place of x as expanding the node x.*

Lemma 7. *If $(S, \overline{S})$ is a minimum cut in a weighted undirected graph G such that no other minimum cut $(A, \overline{A})$ crosses with $(S, \overline{S})$, then $\Lambda(G) = \Lambda(G/S) + \Lambda(G/\overline{S}) - 1$.*

Proof. Note that since $(S, \overline{S})$ is a minimum cut, the value of the minimum cut in G/S and $G/\overline{S}$ will be the same as that in G. First we claim that $\Lambda(G) \leq \Lambda(G/S) + \Lambda(G/\overline{S}) - 1$. This can be seen by observing that corresponding to each minimum cut in G there is a minimum cut in either G/S or $G/\overline{S}$. This follows from the assumption that no minimum cut $(A, \overline{A})$ of G crosses with $(S, \overline{S})$ and so exactly one of the four cases $A \supset S$, $A \supset \overline{S}$, $\overline{A} \supset S$, $\overline{A} \supset \overline{S}$ is true by Lemma 2. Thus the minimum cut $(A, \overline{A})$ remains intact either in G/S or $G/\overline{S}$. Also, $(S, \overline{S})$ appears in both G/S and $G/\overline{S}$ which accounts for subtracting 1. To see $\Lambda(G) \geq \Lambda(G/S) + \Lambda(G/\overline{S}) - 1$, observe that any minimum cut $(A, \overline{A})$ in G/S or in $G/\overline{S}$ has a corresponding minimum cut in G. For example, consider a minimum cut $(A, \overline{A})$ in G/S. Without loss of generality, let the node s in G/S (which corresponds to the contraction of S) be in A. When we expand s, clearly, the minimum cut $(A \bigcup S - \{s\}, \overline{A \bigcup S - \{s\}})$ of G corresponds to the minimum cut $(A, \overline{A})$ of G/S. Moreover, it can be easily verified that the cuts of G which correspond to the cuts of G/S are distinct from the cuts of G which correspond to the cuts of $G/\overline{S}$ except for $(S, \overline{S})$, which is accounted by subtracting 1. Hence the result follows. ■

Lemma 8. *If there are no crossing pairs of minimum cuts in G, then $\Lambda(G) \leq 2n-3$. Moreover, there exists a graph on n nodes, G_n, (for every $n \geq 2$), such that $\Lambda(G_n) = 2n-3$.*

Proof. If $n = 2$, clearly $\Lambda(G) = 1$ and the Lemma is true. Assume that for all graphs with number of nodes $< n$ (where $n \geq 3$), the Lemma is true. Consider a graph G on n nodes with no crossing pairs of minimum cuts. If all the minimum cuts of G are single–node cuts (i.e., of the form $(\{u\}, \overline{\{u\}})$), then clearly there are at most n minimum cuts. Then, $\Lambda(G) \leq n \leq 2n-3$. Otherwise, there is a minimum cut $(S, \overline{S})$ such that $|S| \geq 2$ and $|\overline{S}| \geq 2$. Let $G_1 = G/S$ and $G_2 = G/\overline{S}$. Also, let the number of nodes in G_1 and G_2 be n_1 and n_2 respectively. Since any minimum cut $(A, \overline{A})$ of G does not cross with $(S, \overline{S})$, by Lemma 7, we have $\Lambda(G) = \Lambda(G_1) + \Lambda(G_2) - 1$. Also, it can be easily verified that there will not be any crossing pair of minimum cuts in G_1 or G_2, since such a pair will give rise to a corresponding pair of crossing minimum cuts in G also, which is a contradiction. Thus since G_1 and G_2 have $< n$ vertices, we have $\Lambda(G) \leq 2n_1 - 3 + 2n_2 - 3 - 1 = 2(n+2) - 7 = 2n-3$, since $n_1 + n_2 - 2 = n$. In Theorem 3, we show a way to assign weights to the edges of a clique K_n such that $\Lambda(K_n) = 2n-3$, illustrating the tightness of the bound given by this Lemma. ■

3 The Partition Number and the Structural Parameters

3.1 Partition Number, $p(G)$

Lemma 9. *Let G be a weighted undirected graph. If $(X, \overline{X})$ is a minimum cut of G such that no other minimum cut crosses with $(X, \overline{X})$, then $p(G/X) \leq p(G)$.*

Proof. Suppose $p(G/X) = p' > p(G) = p$. Clearly, $p' \geq 4$ (since by definition of partition number, $p(G) = p \geq 3$). Consider a circular partition $\mathcal{C}' = (U_0, U_1, \cdots, U_{p'-1})$ of G/X. Without loss of generality, assume that the node x obtained by contracting X is present in U_0. We claim that $\mathcal{C} = (W_0, \cdots, W_{p'-1})$ where $W_0 = U_0 - \{x\} \cup X$ and $W_i = U_i$ for $0 < i \leq p'-1$, is a circular partition for G. (This will clearly contradict the assumption that $p(G/X) = p' > p(G)$, proving the Lemma).

Suppose $\mathcal{C} = (W_0, W_1, \cdots, W_{p'-1})$ is not a circular partition for G. Then, by definition of circular partition, there exists a minimum cut $(A, \overline{A})$ of G which crosses with $(W_i, \overline{W}_i)$ for some i.

Case 1 : *$(A, \overline{A})$ does not cross with $(W_0, \overline{W}_0)$ but it crosses with $(W_i, \overline{W}_i)$ for some $i > 0$.* Since $(A, \overline{A})$ does not cross with $(W_0, \overline{W}_0)$, by Lemma 2, we have (1) $A \subseteq W_0$, (2) $\overline{A} \subseteq W_0$, (3) $A \subseteq \overline{W}_0$ or (4) $\overline{A} \subseteq \overline{W}_0$.

> **Case 1.1 :** *$A \subseteq W_0$ or $\overline{A} \subseteq W_0$.* Since $W_0 \subseteq \overline{W}_i$, we have $A \subseteq \overline{W}_i$ or $\overline{A} \subseteq \overline{W}_i$ respectively. So, in both cases $(A, \overline{A})$ does not cross with $(W_i, \overline{W}_i)$ by Lemma 2, contradicting the assumption of Case 1.
>
> **Case 1.2 :** *$A \subseteq \overline{W}_0$ or $\overline{A} \subseteq \overline{W}_0$*, i.e., $W_0 \subseteq \overline{A}$ or $W_0 \subseteq A$ respectively. Since $X \subseteq W_0$, $X \subseteq \overline{A}$ or $X \subseteq A$, which means that all the nodes in X are on the

same side of the cut $(A, \overline{A})$. Thus, the cut $(A', \overline{A'})$ of G/X corresponding to $(A, \overline{A})$ is a minimum cut of G/X. Since $X \subseteq W_0 \subseteq \overline{W}_i$, $X \subseteq \overline{W}_i \cap A$ or $X \subseteq \overline{W}_i \cap \overline{A}$, i.e., all the nodes in X are either in $\overline{W}_i \cap A$ or $\overline{W}_i \cap \overline{A}$. But since $(W_i, \overline{W}_i)$ crosses with $(A, \overline{A})$, the sets $W_i \cap A$, $\overline{W}_i \cap A$, $W_i \cap \overline{A}$ and $\overline{W}_i \cap \overline{A}$ are non-empty. Therefore, clearly when we contract X to get G/X, the corresponding four sets $U_i \cap A'$, $\overline{U}_i \cap A'$, $U_i \cap \overline{A'}$ and $\overline{U}_i \cap \overline{A'}$ are also non-empty. This means that in G/X, $(A', \overline{A'})$ crosses with $(U_i, \overline{U}_i)$ contradicting the assumption that $\mathcal{C}' = (U_0, U_1, \cdots, U_{p'-1})$ is a circular partition for G/X.

Case 2 : $(A, \overline{A})$ *crosses with* $(W_0, \overline{W}_0)$. Remember that by assumption $(A, \overline{A})$ does not cross with $(X, \overline{X})$. We have the following 4 cases by Lemma 2.
(1) $A \subseteq X$, (2) $\overline{A} \subseteq X$, (3) $A \subseteq \overline{X}$ or (4) $\overline{A} \subseteq \overline{X}$.

Case 2.1 : $A \subseteq X$ *or* $\overline{A} \subseteq X$. Since $X \subseteq W_0$, we have $A \subseteq W_0$ or $\overline{A} \subseteq W_0$ respectively, which means by Lemma 2 $(A, \overline{A})$ does not cross with $(W_0, \overline{W}_0)$ in both cases, contradicting the assumption of Case 2.
Case 2.2 : $A \subseteq \overline{X}$ *or* $\overline{A} \subseteq \overline{X}$, i.e., $X \subseteq \overline{A}$ or $X \subseteq A$, which means that all the nodes in X are on the same side of $(A, \overline{A})$. Thus, the cut $(A', \overline{A'})$ of G/X corresponding to $(A, \overline{A})$ is a minimum cut of G/X. Since $X \subseteq W_0$, we have $X \subseteq W_0 \cap A$ or $X \subseteq W_0 \cap \overline{A}$, i.e., all the nodes in X are completely in $W_0 \cap A$ or $W_0 \cap \overline{A}$. But since $(W_0, \overline{W}_0)$ crosses with $(A, \overline{A})$, the sets $W_0 \cap A$, $\overline{W}_0 \cap A$, $W_0 \cap \overline{A}$ and $\overline{W}_0 \cap \overline{A}$ are non-empty. Therefore, clearly when we contract X to get G/X, the corresponding 4 sets $U_0 \cap A'$, $\overline{U}_0 \cap A'$, $U_0 \cap \overline{A'}$ and $\overline{U}_0 \cap \overline{A'}$ are also non-empty. This means that in G/X, $(A', \overline{A'})$ crosses with $(U_0, \overline{U}_0)$ contradicting the assumption that $\mathcal{C}' = (U_0, U_1, \cdots, U_{p'-1})$ is a circular partition for G/X.

Thus, we infer that no minimum cut $(A, \overline{A})$ can cross with any cut $(W_i, \overline{W}_i)$ in the circular partition $\mathcal{C} = (W_0, W_1, \cdots, W_{p'-1})$ for G. But $p(\mathcal{C}) = p' > p = p(G)$, a contradiction. Thus we have $p(G/X) \leq p(G)$. ■

In the following Lemma, we upper–bound $\Lambda(G)$ in terms of the partition number. The tightness of the Lemma will be established in Theorem 8.

Lemma 10. *Let $G = (V, E)$ be a weighted undirected graph, where $|V| = n \geq 2$ and let the partition number $p(G) \leq p$. Then, $\Lambda(G) \leq \frac{(p+1)n}{2} - p$.*

Proof. The proof is by induction on n. If $n = 2$, by definition, $p(G) = 3$ and it is easy to verify that $\Lambda(G) \leq \frac{(p+1)n}{2} - p$. Now, assume that for all graphs with number of nodes $< n$ (where $n \geq 3$), the Lemma is true. Let G be a graph on n nodes ($n \geq 3$) with $p(G) = p$. If $p = 3$, then by Lemma 6, there are no crossing pairs of minimum cuts in G and hence by Lemma 8, $\Lambda(G) \leq 2n - 3 = \frac{(p+1)n}{2} - p$. Now, let $p \geq 4$. By Lemma 6, there exists a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{p-1})$ of G. If $p = n$, then $|U_i| = 1$ for each i and clearly $G = C_n$, a cycle of n nodes with each edge having weight $\frac{\lambda(G)}{2}$. Therefore, it has $\binom{n}{2} = \frac{n(n-1)}{2} = \frac{(n+1)n}{2} - n$ minimum cuts. If $p < n$, then there exists a $U_i \in \mathcal{C}$, such that $|U_i| \geq 2$. Let $G_1 = (V_1, E_1) = G/U_i$ and $G_2 = (V_2, E_2) = G/\overline{U}_i$

be the graphs obtained by contracting U_i and $\overline{U}_i$ respectively. Since $|U_i| \geq 2$ and $|\overline{U}_i| \geq 3$ (note that this follows from $p(G) \geq 4$), clearly, $n > |V_1| \geq 3$ and $n > |V_2| \geq 4$. Let $p_1 = p(G_1)$ and $p_2 = p(G_2)$. Note that by the definition of a circular partition, no minimum cut $(A, \overline{A})$ of G crosses with $(U_i, \overline{U}_i)$. Hence by Lemma 7, we have $\Lambda(G) = \Lambda(G_1) + \Lambda(G_2) - 1$. Now, by the induction assumption, $\Lambda(G_j) \leq \frac{(p_j+1)n_j}{2} - p_j$ (for $j = 1, 2$) and we have,

$$\Lambda(G) \leq \frac{(p_1 + 1)n_1}{2} - p_1 + \frac{(p_2 + 1)n_2}{2} - p_2 - 1 \tag{1}$$

By Lemma 9, we have $p_1 = p(G/U_i) \leq p$ and $p_2 = p(G/\overline{U}_i) \leq p$ since the minimum cut $(U_i, \overline{U}_i)$ does not cross with any other minimum cut in G. For $n_i \geq 2$ and $p_i \leq p$ $(i = 1, 2)$, it is easy to verify that $\frac{(p_i+1)n_i}{2} - p_i \leq \frac{(p+1)n_i}{2} - p$. Substituting in Inequality (1) and noting that $n_1 + n_2 - 2 = n$, we get $\Lambda(G) \leq \frac{(p+1)n}{2} - p$. ■

In the rest of the paper, we show that various structural parameters of a graph can influence the partition number $p(G)$. Thus by means of Lemma 10, we relate the number of minimum cuts, $\Lambda(G)$, with many seemingly unrelated properties of the graph.

Remark : Please note that if $n \geq 2$ and $x \geq p$, then $\frac{(x+1)n}{2} - x \geq \frac{(p+1)n}{2} - p$. In most of the theorems below, we show that p is upper–bounded by a function $f(y)$ of y where y is some parameter of G, thereby showing that $\Lambda(G) \leq \frac{(f(y)+1)n}{2} - f(y)$.

3.2 Radius and Diameter

If $G = (V, E)$ is a connected graph, the *eccentricity* of a node $v \in V$ is defined as $e(v) = \max distance(v, u)$ over all the nodes $u \in V$. The radius of the graph G, $r(G) = \min_{v \in V} e(v)$. A vertex v is a central node if $e(v) = r(G)$. Diameter of G, $d(G) = max_{v \in V} e(v)$. (Note that, in this paper, by "distance", we mean only the distances in the underlying unweighted graph. Thus radius, eccentricity, diameter etc. have nothing to do with the weights).

Theorem 1. *If r is the radius of a weighted undirected graph G, then $\Lambda(G) \leq (r + 1)n - (2r + 1)$ (where $n \geq 2$).*

Proof. Suppose there are no crossing pairs of minimum cuts in G. If follows by Lemma 8 that $\Lambda(G) \leq 2n - 3$. Since the radius is at least 1, it is easy to verify that $\Lambda(G) \leq (r+1)n - (2r+1)$ in this case. Otherwise, by Lemma 6, there exists a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{p-1})$ for G, where $p = p(G) \geq 4$. Let $x \in U_i$ be a central node of G. Let $y \in U_{i + \lfloor \frac{p}{2} \rfloor \ mod \ p}$. Clearly, $distance(x, y) \geq \lfloor \frac{p}{2} \rfloor$. That is, $r \geq \lfloor \frac{p}{2} \rfloor$ or $p \leq 2r + 1$. Now, by Lemma 10, we get $\Lambda(G) \leq (r+1)n - (2r+1)$. ■

We note that the bound given by the above Theorem can be tight. For example, consider C_{2n+1}, the cycle on $2n+1$ nodes. Clearly, the radius of C_{2n+1} is n and the number of minimum cuts $= \binom{2n+1}{2} = (n+1)(2n+1) - (2n+1)$.

Observe that similar arguments as given for the case of radius hold good for the diameter also. Thus,

$$\Lambda(G) \leq (d+1)n - (2d+1)$$

This can also be verified from $\Lambda(G) \leq (r+1)n - (2r+1) \leq (d+1)n - (2d+1)$ by noting that $d \geq r$ and $n \geq 2$.

3.3 Politician node

An interesting special case of Theorem 1 is when $radius(G) = 1$. Then, there exists a node which is adjacent to every other node of the graph. (Such a node is called a *politician* node). Thus, if there is a politician node in the graph, then $\Lambda(G) \leq 2n-3$, by Theorem 1. In fact more is true.

Theorem 2. *If there is a politician node u in G, then there cannot be any crossing pairs of minimum cuts in G.*

Proof. If there is a crossing pair of minimum cuts, then by Lemma 6, there is a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{k-1})$ $(k \geq 4)$. Without loss of generality let $u \in U_0$. Clearly, u cannot be adjacent to any node in U_2, by the definition of circular partition, contradicting the assumption that u is a politician node. ∎

Note that in a complete graph, K_n, every node is a politician node. Thus, there are no crossing pairs of minimum cuts in a clique. Below, we show a way to assign weights to the edges of K_n such that the number of minimum cuts $\Lambda(K_n) = 2n-3$, thus illustrating that the bound of Lemma 8 is tight. Moreover, since radius of a clique is 1, this is a tight example for Theorem 1 too. Since a complete graph is a chordal graph, the example below illustrates the tightness of Theorem 7 also.

Theorem 3. *For any $n \geq 2$ and $\lambda > 0$, there exists a weighted complete graph $\mathcal{K}_n$ such that $\lambda(\mathcal{K}_n) = \lambda$ and $\Lambda(\mathcal{K}_n) = 2n-3$. Moreover every node x of $\mathcal{K}_n$ defines a minimum cut $(\{x\}, \overline{\{x\}})$ of $\mathcal{K}_n$.*

Proof. For $n = 2$, it is trivial. For $n = 3$, let $\mathcal{K}_3$ be the triangle with each edge having weight $\frac{\lambda}{2}$. Clearly, $\Lambda(\mathcal{K}_3) = 2.3 - 3 = 3$. Also, note that every node x in $\mathcal{K}_3$ defines a minimum cut $(\{x\}, \overline{\{x\}})$. Now inductively assume that there exists a weighted complete graph on $n-1$ nodes $\mathcal{K}_{n-1}$ $(n \geq 4)$, such that $\lambda(\mathcal{K}_{n-1}) = \lambda$, $\Lambda(\mathcal{K}_{n-1}) = 2n-5$ and every node x of $\mathcal{K}_{n-1}$ defines a minimum cut $(\{x\}, \overline{\{x\}})$ in $\mathcal{K}_{n-1}$. We show how to construct a weighted complete graph $\mathcal{K}_n$ from $\mathcal{K}_{n-1}$, such that $\lambda(\mathcal{K}_n) = \lambda$, $\Lambda(\mathcal{K}_n) = 2n-3$ and every node x of $\mathcal{K}_n$ defines a minimum cut $(\{x\}, \overline{\{x\}})$ in $\mathcal{K}_n$.

Let u be any node of $\mathcal{K}_{n-1}$. We remove u from $\mathcal{K}_{n-1}$ (along with the edges incident on it) and then add two other nodes u', u'' in its place. From each node y in $\mathcal{K}_{n-1}$, $(y \neq u)$, we add the edges (y, u') as well as (y, u'') to the new nodes and assign weights $w(y,u') = w(y,u'') = \frac{w(y,u)}{2}$. Also add the edge (u',u'') with $w(u',u'') = \frac{\lambda}{2}$.

It is easy to see that the new graph $\mathcal{K}_n$ is a complete graph. Let $S = \{u',u''\}$. Since $(\{u\},\overline{\{u\}})$ is a minimum cut of $\mathcal{K}_{n-1}$, $w(S,\overline{S}) = \lambda(\mathcal{K}_{n-1}) = \lambda$. We claim that $\lambda(\mathcal{K}_n) = \lambda$. If not, there exists a cut $(A,\overline{A})$ in $\mathcal{K}_n$ such that $w(A,\overline{A}) < \lambda$. If both the nodes of $S = \{u',u''\}$ are present on the same side of the cut $(A,\overline{A})$, then the corresponding cut in $\mathcal{K}_{n-1}$ obtained by contracting S will also have weight $< \lambda$, contradicting the assumption that $\lambda(\mathcal{K}_{n-1}) = \lambda$. Therefore, without loss of generality, we can assume that $u' \in A$ and $u'' \in \overline{A}$. Then clearly, $(u',u'') \in E(A,\overline{A})$. Also, for each y in $\mathcal{K}_n$, ($y \neq u'$ and $y \neq u''$), exactly one of the edges (u',y) or (u'',y) belongs to $E(A,\overline{A})$. Now recall that $w(y,u') = w(y,u'') = \frac{w(y,u)}{2}$ and $\sum_{y\neq u} w(y,u) = \lambda$ in $\mathcal{K}_{n-1}$. So we have $w(A,\overline{A}) \geq w(u',u'') + \frac{1}{2}\sum_{y\neq u',u''}(w(y,u') + w(y,u'')) = \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$, which is a contradiction to the assumption that $w(A,\overline{A}) < \lambda$. Hence, $\lambda(\mathcal{K}_n) = \lambda$ and $(S,\overline{S})$ is a minimum cut of $\mathcal{K}_n$. Also, since $\mathcal{K}_n$ is a clique, by Theorem 2, no minimum cut $(A,\overline{A})$ of $\mathcal{K}_n$ crosses with $(S,\overline{S})$. By Lemma 7, we have $\Lambda(\mathcal{K}_n) = \Lambda(\mathcal{K}_n/S) + \Lambda(\mathcal{K}_n/\overline{S}) - 1$. But, $\mathcal{K}_n/S = \mathcal{K}_{n-1}$ and $\mathcal{K}_n/\overline{S} = \mathcal{K}_3$. Thus we have $\Lambda(\mathcal{K}_n) = 2n-5+3-1 = 2n-3$. Also it is easy to verify that every node x of $\mathcal{K}_n$ defines a minimum cut $(\{x\},\overline{\{x\}})$ of $\mathcal{K}_n$. ■

3.4 Maximum and Minimum Degree

The maximum degree $\Delta(G)$ (when it is reasonably high) can also constrain the number of minimum cuts $\Lambda(G)$.

Theorem 4. *If Δ is the maximum degree of a weighted undirected graph G, then $\Lambda(G) \leq \frac{(n-\Delta+3)n}{2} - (n-\Delta+2)$, where $n \geq 2$.*

Proof. Suppose there is no crossing pairs of minimum cuts in G, then by Lemma 8, $\Lambda(G) \leq 2n-3 \leq \frac{(n-\Delta+3)n}{2} - (n-\Delta+2)$, which will be true if $0 \leq n^2-(\Delta+3)n+(2\Delta+2)$ or $0 \leq (n-\Delta-1)(n-2)$ which is true since $n \geq 2$ and $\Delta \leq n-1$. Now, if there is a crossing pair of minimum cuts in G, then by Lemma 6, there is a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{p-1})$ ($p = p(G) \geq 4$). Without loss of generality , let the maximum degree node $u \in U_1$. Then, $|U_0 \cup U_1 \cup U_2| \geq \Delta + 1$ since every neighbour of u must be in U_0, U_1 or U_2. Thus, $p \leq 3 + (n-\Delta-1) = n-\Delta+2$ since each U_i of the circular partition must contain at least 1 node. By Lemma 10, $\Lambda(G) \leq \frac{(n-\Delta+3)n}{2} - (n-\Delta+2)$. ■

Interestingly, the minimum degree of the graph can also control the number of minimum cuts.

Theorem 5. *If δ is the minimum degree of a weighted undirected graph G, then $\Lambda(G) \leq (\frac{3n}{2(\delta+1)} + 1.5)n - (\frac{3n}{\delta+1} + 2)$, where $n \geq 2$.*

Proof. If there are no crossing pairs of minimum cuts in G, it can be easily verified that $\Lambda(G) \leq 2n-3 \leq (\frac{3n}{2(\delta+1)} + 1.5)n - (\frac{3n}{\delta+1} + 2)$ for $n \geq 2$. Otherwise consider a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{p-1})$ ($p = p(G) \geq 4$). Group the subsets in $\mathcal{C}$ into $\lfloor \frac{p}{3} \rfloor$ triplets $(U_{3i}, U_{3i+1}, U_{3i+2})$ for $0 \leq i \leq \lfloor \frac{p}{3} \rfloor - 1$. $|U_{3i}| + |U_{3i+1}| + |U_{3i+2}| \geq \delta + 1$ since each neighbour of a node $u \in U_{3i+1}$ must be in one of the three sets in the corresponding triplet. Thus, $\lfloor \frac{p}{3} \rfloor (\delta + 1) \leq n$ and the result follows by Lemma 10. ■

3.5 Chordality

Recall that the chordality of a graph is the length of the longest induced cycle in the graph. We upper–bound $\Lambda(G)$ in terms of chordality in the following Theorem. Its tightness is established in Theorem 8.

Theorem 6. *If G is a weighted undirected graph with chordality k, then $\Lambda(G) \leq \frac{(k+1)n}{2} - k$, where $n \geq 2$.*

Proof. If there are no crossing pairs of minimum cuts in G, then by Lemma 8, $\Lambda(G) \leq 2n - 3 \leq \frac{(k+1)n}{2} - k$, since k is at least 3 by definition and $n \geq 2$. Otherwise, consider a circular partition $\mathcal{C}$ for G such that $p(\mathcal{C}) = p(G)$. If $p(\mathcal{C}) > k$, clearly there is an induced cycle in G with length $> k$, contradicting the k-chordality of G. It follows that $p(G) \leq k$. Therefore, by Lemma 10, $\Lambda(G) \leq \frac{(k+1)n}{2} - k$. ■

Theorem 7. *If G is a weighted chordal graph, then $\Lambda(G) \leq 2n-3$, where $n \geq 2$. Moreover there are no crossing pairs of minimum cuts in G. Also, there exists a weighted chordal graph G, for every $n \geq 2$, such that $\Lambda(G) = 2n - 3$.*

Proof. Since for chordal graphs $k = 3$ (by definition), $\Lambda(G) \leq 2n - 3$ follows from Theorem 6. If there is a crossing pair of minimum cuts in G, then there is a circular partition $\mathcal{C}$ for G with $p(\mathcal{C}) \geq 4$ by Lemma 6. This immediately implies an induced cycle of length ≥ 4, contradicting the fact that G is chordal. Finally since complete graphs are chordal graphs, the construction of Theorem 3, establishes the tightness of this bound. ■

3.6 A Tight Construction

We establish the tightness of Theorem 6 and Lemma 10, by the following construction.

Theorem 8. *For $k \geq 3$ and $\lambda > 0$, there exists a family $\mathcal{G}$ of weighted undirected k-chordal graphs such that for each graph $G_n \in \mathcal{G}$ with n nodes, $\Lambda(G_n) = \frac{(k+1)n}{2} - k$, weight of the minimum cut $= \lambda$ and $p(G_n) = k$. Moreover, every node u of G_n defines a minimum cut $(\{u\}, \overline{\{u\}})$.*

Proof. When $k = 3$, the family of cliques constructed in Theorem 3 has the desired properties, i.e., $\lambda(\mathcal{K}_n) = \lambda$, $\Lambda(\mathcal{K}_n) = 2n - 3$, $p(\mathcal{K}_n) = 3$ and every node x of $\mathcal{K}_n$ defines a minimum cut $(\{x\}, \overline{\{x\}})$. In the rest of the proof we assume that $k \geq 4$. First note that $G_k = C_k$, the cycle on k nodes with each edge of weight $\frac{\lambda}{2}$ is a k-chordal graph with the desired properties, i.e., $\lambda(C_k) = \lambda$,

$\Lambda(C_k) = \binom{k}{2} = \frac{(k+1)k}{2} - k$ and $p(C_k) = k$. Also every node $x \in C_k$ defines a minimum cut.

Now we show how to inductively construct the desired family. Let $G_n = (V, E)$ be a k-chordal graph on n nodes such that $\Lambda(G_n) = \frac{(k+1)n}{2} - k$, $p(G_n) = k$ and $\lambda(G_n) = \lambda$. Also assume that each node x in G_n defines a minimum cut. We describe how to construct a k-chordal graph $G_{n'} = (V', E')$ from G_n where $|V'| = n' = n + k - 2$, such that $\Lambda(G_{n'}) = \frac{(k+1)n'}{2} - k$, $\lambda(G_{n'}) = \lambda$, $p(G_{n'}) = k$ and every node of $G_{n'}$ defines a minimum cut, thereby proving the existence of the desired family.

Construction of $G_{n'}$ from G_n : Let u be any node in G_n. Then, let $V' = V - \{u\} \cup P$, where $P = \{y_1, y_2, \cdots, y_{k-1}\}$ are not already present in V. Let $N(u) = \{z_1, z_2, \cdots, z_l\}$ be the neighbours of u in G_n. Let $E' = E - \{(u, z_i) : 1 \le i \le l\} \cup \{(y_1, z_i) : 1 \le i \le l\} \cup \{(y_{k-1}, z_i) : 1 \le i \le l\} \cup \{(y_j, y_{j+1}) : 1 \le j \le k-2\}$ where the weights $w(z_i, y_1) = w(z_i, y_{k-1}) = \frac{w(z_i,u)}{2}$ for $1 \le i \le l$ and $w(y_j, y_{j+1}) = \frac{\lambda}{2}$ for $1 \le j \le k-2$. Thus, to get $G_{n'}$, we remove u from G_n along with the edges incident on it and add a path $(y_1, y_2, \cdots, y_{k-1})$ with each edge of weight $\frac{\lambda}{2}$. Also each neighbour z_i of u in G_n is now connected to y_1 and y_{k-1}. Moreover, the weight of (z_i, y_1) and (z_i, y_{k-1}) will be assigned half the weight of the edge (z_i, u) in G_n. It may be noted that the contracted graphs $G_{n'}/P = G_n$ and $G_{n'}/\overline{P} = C_k$ with each edge having weight $\frac{\lambda}{2}$.

Claim 0 : Let $(S, \overline{S})$ be a minimum cut of $G_{n'}$ which crosses with the cut $(P, \overline{P})$, where $P = \{y_1, y_2, \cdots, y_{k-1}\}$. Then, exactly one of the two edges (z_i, y_1) or (z_i, y_{k-1}) ($1 \le i \le l$), will belong to $E(S, \overline{S})$. (Recall that $\{z_1, z_2, \cdots, z_l\}$ are the nodes in $G_{n'}$ which correspond to the neighbours of u in G_n).

Proof. First, we claim that both the nodes y_1 and y_{k-1} cannot be on the same side of the minimum cut $(S, \overline{S})$. Suppose for example, $\{y_1, y_{k-1}\} \subseteq S$. Because all the edges from P to $\overline{P}$ is incident on either y_1 or y_{k-1}, $E(\overline{S} \cap P, \overline{S} \cap \overline{P}) = \emptyset$. (Note that $\overline{S} \cap P$ and $\overline{S} \cap \overline{P}$ will be non empty since $(S, \overline{S})$ is assumed to cross with $(P, \overline{P})$). Therefore the induced subgraph on $\overline{S}$ will be disconnected, which is a contradiction by Lemma 1, since $(S, \overline{S})$ is assumed to be a minimum cut. Now without loss of generality assume that $y_1 \in S$ and $y_{k-1} \in \overline{S}$. Then clearly one of the two edges (z_i, y_1) or (z_i, y_{k-1}) (since both these edges exist by construction), will belong to $E(S, \overline{S})$.

Claim 1 : $\lambda(G_{n'}) = \lambda(G_n) = \lambda$ and $(P, \overline{P})$ is a minimum cut of $G_{n'}$.

Proof. First note that the cut $(P, \overline{P})$ in $G_{n'}$ has weight $w(P, \overline{P}) = \lambda$. This is easily seen from the fact that if we contract P, replacing the set P with the node u, we will get G_n (i.e., $G_{n'}/P = G_n$) and the cut $(P, \overline{P})$ in $G_{n'}$ will correspond to the single vertex minimum cut $(\{u\}, \overline{\{u\}})$ in G_n. Now we will show that every cut in $G_{n'}$ has weight at least λ, thereby establishing that $\lambda(G_{n'}) = \lambda$ and $(P, \overline{P})$ is a minimum cut of $G_{n'}$. Suppose $\lambda(G_{n'}) < \lambda$. Then let $(S, \overline{S})$ be a minimum cut of $G_{n'}$. If S (or $\overline{S}$) is a subset of P or $\overline{P}$, then one of the contracted graphs

$G_{n'}/P = G_n$ or $G_{n'}/\overline{P} = C_k$ will contain a corresponding cut with the same value, which clearly will be a contradiction since $\lambda(G_n) = \lambda$ and $\lambda(C_k) = \lambda$. Thus by Lemma 2, $(S, \overline{S})$ must cross with $(P, \overline{P})$ in $G_{n'}$, which means $S \cap P, S \cap \overline{P}, \overline{S} \cap P$ and $\overline{S} \cap \overline{P}$ are non empty. Now, by Claim 0, exactly one of the two edges (z_i, y_1) or (z_i, y_{k-1}) ($1 \leq i \leq l$), will belong to $E(S, \overline{S})$. Recall that $w(z_i, y_1) = w(z_i, y_{k-1})$ and $\sum_{i=1}^{i=l} w(z_i, y_1) + \sum_{i=1}^{i=l} w(z_i, y_{k-1}) = w(\{u\}, \overline{\{u\}}) = \lambda(G_n) = \lambda$. Therefore, the total contribution to the weight of $(S, \overline{S})$ due to the edges of the form (y_j, z_i) ($j = 1, k-1$ and $1 \leq i \leq l$) is $\frac{\lambda}{2}$. Also, since each edge in the path defined by the nodes in P has weight $\frac{\lambda}{2}$, it is clear that $w(S \cap P, \overline{S} \cap P) \geq \frac{\lambda}{2}$. Thus, considering both the contributions, we infer that $w(S, \overline{S}) \geq \lambda$, contradicting the assumption that $w(S, \overline{S}) < \lambda$. So we have established that $\lambda(G_{n'}) = \lambda$, and therefore $(P, \overline{P})$ is a minimum cut of $G_{n'}$.

Claim 2 : No minimum cut $(S, \overline{S})$ of $G_{n'}$ crosses with the minimum cut $(P, \overline{P})$ and $\Lambda(G_{n'}) = \frac{(k+1)n'}{2} - k$.

Proof. Suppose a minimum cut $(S, \overline{S})$ crosses with the minimum cut $(P, \overline{P})$ in $G_{n'}$. Then by Claim 0, exactly one of the two edges (z_i, y_1) or (z_i, y_{k-1}) ($1 \leq i \leq l$), will belong to $E(S, \overline{S})$. Clearly both (z_i, y_1) and (z_i, y_{k-1}) are in $E(P, \overline{P})$. Thus $E(S, \overline{S}) \cap E(P, \overline{P}) \neq \emptyset$, contradicting Lemma 4. Therefore we infer that no minimum cut $(S, \overline{S})$ of $G_{n'}$ can cross with $(P, \overline{P})$. Now by applying Lemma 7 and noting that $n' = n + k - 2$, we have

$$\begin{aligned} \Lambda(G_{n'}) &= \Lambda(G_n) + \Lambda(C_k) - 1 \\ &= \frac{(k+1)n}{2} - k + \frac{k(k-1)}{2} - 1 \\ &= \frac{(k+1)(n+k-2)}{2} - k \\ &= \frac{(k+1)n'}{2} - k \end{aligned}$$

Claim 3 : Each node $x \in V'$ defines a minimum cut $(\{x\}, \overline{\{x\}})$ of $G_{n'}$.

Proof. It is easy to check that the sum of weights on edges incident on the nodes of $\overline{P}$ has not changed from what it was in G_n. Also, it is clear that for $2 \leq i \leq k-2$, $w(\{y_i\}, \overline{\{y_i\}}) = \lambda$. Finally $w(\{y_1\}, \overline{\{y_1\}}) = w(y_1, y_2) + \sum_{i=1}^{i=l} w(y_1, z_i)$ $= \frac{\lambda}{2} + \frac{w(\{u\}, \overline{\{u\}})}{2} = \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$. The same argument hold for y_{k-1} also.

Claim 4 : $G_{n'}$ is k-chordal.

Proof. Suppose there is an induced cycle C of length $> k$ in $G_{n'}$. We consider two cases, and show contradictions in both cases.

Case 1 : C contains a node y_i from P other than y_1 or y_{k-1}. In this case, clearly C must contain the nodes y_1 and y_{k-1} also. Let z_i be the neighbour of y_1 in C from $\overline{P}$. Then z_i must be the neighbour of y_{k-1} also in C – since otherwise the edge (z_i, y_{k-1}) will form a chord for C. (Note that this edge exists by construction of $G_{n'}$). Thus C will be $(z_i, y_1, y_2, \cdots, y_{k-1}, z_i)$, a cycle of length k, contradicting the assumption that $|C| > k$.

Case 2 : C does not contain any node y_i from P other than y_1 or y_{k-1}. In this case clearly C is a (induced) subgraph of $G(\overline{P} \bigcup \{y_1, y_{k-1}\})$. We claim that C must contain both the nodes y_1 and y_{k-1}. Otherwise, if for example C does not contain y_1, $|C| > k$ cannot be true, since the structure of $G(\overline{P} \bigcup \{y_{k-1}\})$ is the same as that of G_n (except for the weights) and G_n is assumed to be k–chordal. Thus we infer that $\{y_1, y_{k-1}\} \subset C$. Now let z_i and z_j be the neighbours of y_1 in the cycle C. (Note that $z_i \neq y_{k-1}$ and $z_j \neq y_{k-1}$ since y_1 and y_{k-1} are not adjacent in $G_{n'}$). Then z_i and z_j must be the neighbours of y_{k-1} also, in C. Otherwise, for example if z_i is not a neighbour of y_{k-1} in C, clearly the edge (z_i, y_{k-1}) (which exists by construction) will form a chord for C, contradicting the fact that C is a chordless cycle. Now if z_i and z_j are the neighbours of both y_1 and y_{k-1} in C, clearly $C = (y_1, z_i, y_{k-1}, z_j, y_1)$, a cycle of length four, contradicting the assumption that $|C| > k$, since $k \geq 4$ by assumption.

Claim 5 : $p(G_{n'}) = k$.

Proof. By Claim 4, the chordality of $G_{n'}$ is k. Since the partition number is upper–bounded by the chordality (see the proof of Theorem 6), we have $p(G_{n'}) \leq k$. Now, since by Claim 2, the minimum cut $(P, \overline{P})$ does not cross with any other minimum cut in $G_{n'}$, applying Lemma 9 and the induction assumption that $p(G_n) = k$, we get $k = p(G_n) = p(G_{n'}/P) \leq p(G_{n'}) \leq k$. Therefore, $p(G_{n'}) = k$. ■

4 Edge Expansion and Girth

In this section we upper–bound $\Lambda(G)$ in terms of the ratio $\frac{\lambda}{\beta}$, where $\lambda = \lambda(G)$ and $\beta = \beta_w(G)$ is the weighted edge expansion of G. It can be easily verified that $1 \leq \frac{\lambda}{\beta} \leq \frac{n}{2}$. Our bound works for $\frac{\lambda}{\beta} \in [1, \frac{n}{3})$.

Lemma 11. *If G is a weighted undirected graph with weighted edge expansion $\beta_w(G) = \beta$, then for any minimum cut $(S, \overline{S})$, $|S| > \frac{n}{2}$ or $|S| \leq \left\lfloor \frac{\lambda}{\beta} \right\rfloor$.*

Proof. Suppose $|S| \leq \frac{n}{2}$. Then, $w(S, \overline{S}) = \lambda \geq \beta |S|$ by definition of weighted edge expansion. Since, $|S|$ is an integer, we have $|S| \leq \left\lfloor \frac{\lambda}{\beta} \right\rfloor$. ■

Theorem 9. *Let G be a weighted undirected graph with $\lambda(G) = \lambda$ and the weighted edge expansion $\beta_w(G) = \beta$. If $1 \leq \left\lfloor \frac{\lambda}{\beta} \right\rfloor < \frac{n}{3}$, then $\Lambda(G) \leq \frac{(\left\lfloor \frac{\lambda}{\beta} \right\rfloor + 3)}{2} n - (\left\lfloor \frac{\lambda}{\beta} \right\rfloor + 2)$.*

Proof. Let $\gamma = \left\lfloor \frac{\lambda}{\beta} \right\rfloor$. Suppose there are no crossing pairs of minimum cuts in G. Then by Lemma 8, $\Lambda(G) \leq 2n - 3 \leq \frac{(\gamma+3)}{2} n - (\gamma + 2)$, since $\gamma \geq 1$. Otherwise, by Lemma 6, there exists a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{p-1})$ of G with $p = p(G) \geq 4$. We claim that $p \leq \gamma + 2$. Suppose $p > \gamma + 2$. We will show a contradiction. Let U_j be the subset in $\mathcal{C}$ such that $|U_j| = \max_i |U_i|$. We will show first that $|U_i| \leq \gamma$, for all $i \in \{(j+1)\, mod\, p, (j+2)\, mod\, p, \cdots, (j+\gamma+1)\, mod\, p\}$. If $|U_j| \leq \gamma$, this is clearly true since $|U_j| = \max_i |U_i|$ by assumption. Remembering that $(U_j, \overline{U}_j)$ is a minimum cut, by Lemma 11, if $|U_j| > \gamma$, then $|U_j| > \frac{n}{2}$. Therefore, $|V - U_j| < \frac{n}{2}$. Thus, for $i \in \{(j+1)\ mod\ p, (j+2)\ mod\ p, \cdots, (j+\gamma+1)\ mod\ p\}$, $|U_i| \leq |V - U_j| < \frac{n}{2}$ and since $(U_i, \overline{U}_i)$ is a minimum cut, by Lemma 11, $|U_i| \leq \gamma$, as required.

Now let k be the smallest integer such that $|\bigcup_{i=j+1}^{j+k} U_i| > \gamma$. (Note that $\gamma + 1 \geq k \geq 2$). Let $X = \bigcup_{i=j+1}^{j+k} U_i$. Note that since $|X - U_{j+k}| \leq \gamma$ and $|U_{j+k}| \leq \gamma$, we have

$$|X| \leq 2\gamma < \frac{2n}{3} \tag{2}$$

But $(X, \overline{X})$ is a minimum cut by Lemma 5. Since $|X| > \gamma$, we have by Lemma 11, $|X| > \frac{n}{2}$. That is, $|\overline{X}| < \frac{n}{2}$. It follows from the Inequality 2 that $\gamma < \frac{n}{3} < |\overline{X}| < \frac{n}{2}$, which is again a contradiction by Lemma 11 since $(\overline{X}, X)$ is a minimum cut. Thus we infer that $p \leq \gamma + 2$ and hence by Lemma 10, the theorem follows. ■

The observation that the edge expansion can control the number of minimum cuts allows us to obtain a upper bound for $\Lambda(G)$ in terms of the girth in the case of *unweighted graphs.* The following classical results are not very difficult to prove.

Lemma 12. *If $(S, \overline{S})$ is a minimum cut of an unweighted undirected graph G, then $|S| = 1$ or $|S| \geq \delta$, where δ is the minimum degree of G.*

Lemma 13 (See Bollabas [3], page 126.). *Let g be the girth of an unweighted undirected graph G, with $m \geq 2n$. Then, $g \leq 2\left\lceil \log_{\lfloor \frac{\sigma}{2} \rfloor} n \right\rceil + 1$, where $\sigma = \frac{2m}{n}$, is the average degree. (m is the number of edges of G).*

Lemma 14 (See Harary [9]). *If δ is the minimum degree and $\lambda(G) = \lambda$ be the size of a minimum cut (i.e., edge connectivity) in an undirected unweighted graph G, then $\delta \geq \lambda$.*

Theorem 10. *If G is an undirected unweighted graph with minimum degree δ (at least 6) and girth g (at least 4), then $\Lambda(G) < (\frac{n}{x+1} + 1)n - (\frac{2n}{x+1} + 1)$, where $x = \left(\frac{\delta-3}{2}\right)^{\frac{g-3}{2}}$.*

Proof. Consider $G(X)$, the induced subgraph on any subset $X \subset V$, such that $|X| \leq \left(\frac{\delta-3}{2}\right)^{\frac{g-3}{2}}$. Since the girth of $G(X)$ is at least g (the girth of G), by Lemma 13, $g \leq 2\left\lceil \log_{\lfloor \frac{\sigma}{2} \rfloor} |X| \right\rceil + 1$, where σ is the average degree of the induced subgraph $G(X)$. (The Lemma is applicable since $\delta \geq 6$ and so $m \geq 3n > 2n$).

$$g \leq 2(\log_{\lfloor \frac{\sigma}{2} \rfloor} |X| + 1) + 1 \leq 2 \log_{\lfloor \frac{\sigma}{2} \rfloor} \left(\frac{\delta - 3}{2}\right)^{\frac{g-3}{2}} + 3$$

After rearranging,

$$1 \leq \log_{\lfloor \frac{\sigma}{2} \rfloor} \left(\frac{\delta - 3}{2} \right)$$
$$\frac{\delta - 3}{2} \geq \left\lfloor \frac{\sigma}{2} \right\rfloor > \frac{\sigma}{2} - 1$$

So, $\delta - \sigma > 1$. Thus, $\frac{|E(X,\overline{X})|}{|X|} \geq \frac{\delta|X| - \sigma|X|}{|X|} = \delta - \sigma > 1$. Hence, for all subsets X with $|X| \leq x = \left(\frac{\delta-3}{2}\right)^{\frac{g-3}{2}}$, we have $|E(X,\overline{X})| > |X|$. Now, consider a minimum cut $(S,\overline{S})$ with $|S| \leq |\overline{S}|$. We claim that $|S| = 1$ or $|S| > x$. Suppose not. Then by Lemma 12, $x \geq |S| \geq \delta$ and also by Lemma 14, $\delta \geq \lambda$. Thus, $\delta \geq \lambda = |E(S,\overline{S})| > |S| \geq \delta$, a contradiction.

Suppose there is a circular partition $\mathcal{C} = (U_0, U_1, \cdots, U_{p-1})$ for G. By the above observation, $|U_i| = 1$ or $|U_i| > x$. If $|U_i| = 1$, then we claim that $|U_{i-1 \ mod \ p}| > x$ and $|U_{i+1 \ mod \ p}| > x$. This is because, if $|U_i| = 1$ (i.e., $(U_i, \overline{U}_i)$ defines a single-node minimum cut), then size of the minimum cut $\lambda = \delta$. Now by the definition of circular partition, $|E(U_{i+1 \ mod \ p}, U_i)| = |E(U_{i-1 \ mod \ p}, U_i)| = \frac{\lambda}{2} = \frac{\delta}{2} \geq 3$, since $\delta \geq 6$. Thus, $|U_{i-1 \ mod \ p}| > 1$ and $|U_{i+1 \ mod \ p}| > 1$ since G is an *unweighted simple* graph and it follows that $|U_{i-1 \ mod \ p}| > x$ and $|U_{i+1 \ mod \ p}| > x$. Now, for each $0 \leq i \leq \lfloor \frac{p}{2} \rfloor - 1$, $|U_{2i} \bigcup U_{2i+1}| > x+1$. Therefore, $\lfloor \frac{p}{2} \rfloor (x+1) < n$ and so, $p < \frac{2n}{x+1} + 1$ and the Theorem follows by Lemma 10. ■

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