Free vibration analysis of rotating tapered blades using Fourier-*p* superelement

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Abstract. A numerically efficient superelement is proposed as a low degree of freedom model for dynamic analysis of rotating tapered beams. The element uses a combination of polynomials and trigonometric functions as shape functions in what is also called the Fourier-*p* approach. Only a single element is needed to obtain good modal frequency prediction with the analysis and assembly time being considerably less than for conventional elements. The superelement also allows an easy incorporation of polynomial variations of mass and stiffness properties typically used to model helicopter and wind turbine blades. Comparable results are obtained using one superelement with only 14 degrees of freedom compared to 50 conventional finite elements with cubic shape functions with a total of 100 degrees of freedom for a rotating cantilever beam. Excellent agreement is also shown with results from the published literature for uniform and tapered beams with cantilever and hinged boundary conditions. The element developed in this work can be used to model rotating beam substructures as a part of complete finite element model of helicopters and wind turbines.

Keywords: rotating beams; superelement; free vibration; finite element method; helicopter blades; wind turbine blades.

1. Introduction

Rotating blades are important structural members of wind turbines, steam and gas turbines, helicopter rotors and aircraft propellers. These blades are often idealized as rotating beams. Prediction of the natural frequencies of such blades is important because of the design requirement of keeping the natural frequencies away from multiples of the rotor speed and for dynamic analysis (Hosseini and Khadem 2005, Al-Qaisia and Al-Bedoor 2005, Lin *et al*. 2004, Munteanu *et al*. 2004, Lee *et al*. 2004, Furta 2003, Chandiramani *et al*. 2002, Hu *et al*. 2002, Yoo *et al*. 2002, Datta and Ganguli 1990). For a relatively long helicopter rotor blade, the simple and accurate representation is the Euler-Bernoulli beam model. A helicopter rotor blade can undergo out-of-plane bending, in-plane bending and torsion. Due to the centrifugal stiffening effect, the vibration characteristics of rotating Euler-Bernoulli beams vary significantly from those of non-rotating beams and need numerical methods such as Galerkin, Ritz or finite element methods for solution (Zhao and Dewolf

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2007, Al-Sadder *et al*. 2006, Lin and Tsai 2006, Tufekci and Arpaci 2006, Lin and Tsai 2005). Conventional finite element methods (CFEM) have advantages over the Galerkin and Ritz methods since they can be easily modified for any boundary conditions and such methods are often used for rotating beam problems. In finite element development studies, it is typical that couplings are ignored and transverse bending vibration is studied as mentioned by Wang and Wereley (2004). However, the blades are non-uniform and both mass and flexural stiffness are represented as polynomials which is often done in rotor blade dynamic analysis. Wang and Wereley (2004) show that wind turbine blades are well modeled by linear mass and stiffness distribution and helicopter blades by linear mass and quartic stiffness distribution.

Typically, the conventional finite element method (CFEM) for rotating beams uses cubic polynomials as interpolating functions and convergence is achieved by increasing the number of elements. Since dynamic analysis requires at least the first five modes, capturing these modes accurately can need many elements which leads to a large size eigenvalue problem. The resulting large-degree-of-freedom FEM model is impractical in a real-time dynamic simulation or control problems for which finite element models are often used (Cai *et al*. 2004, Yang *et al*. 2004, Fung *et al*. 2004, Khulief 2001, Thakkar and Ganguli 2004, Thakkar and Ganguli 2006). As a result, the model order must be reduced via static or dynamic procedures to a practical number of degrees of freedom, which is constrained by simulation time, or the control interval in a digital control system (2004). Furthermore, the use of many elements requires careful development of the finite element mesh for non-uniform rotor blades and increase in computation time due to assembly (Ganguli *et al*. 1998).

To address the shortcomings of CFEM relating to large number of elements and the consequently large size eigenvalue problem, another approach of FEM called the spectral finite element method (SFEM, also called dynamic stiffness method) has evolved for obtaining the same accuracy using fewer number of elements (Wang and Wereley 2004, Banerjee 2000, Wright *et al*. 1982). Banerjee (2000) considered uniform beams and Wang and Wereley (2004) extended the concept to tapered beams. Banerjee used many elements to model a tapered beam whereas Wang and Wereley developed a tapered element and used only one spectral element. However, a very large number of terms in the power series solution were needed. For example, Wang and Wereley used as many as 350 terms in the Frobenius power series method to find the frequencies of non-uniform rotating beams (2004). In the SFEM, the shape functions are duplicated from exact wave propagation solutions using the governing equation (Vinod *et al*. 2006, Vinod *et al*. ????). However, since SFEM uses the solution obtained in frequency domain, the natural frequencies are obtained by solving transcendental equations instead of solving the eigenvalue problems as in CFEM, which can be quite complicated.

Since practical beams are non-uniform, it is advantageous to develop accurate finite element models for them using a minimum number of finite elements. This could be done by constructing a superelement with accurate interpolating functions. Using a single element leads to easy handling of the polynomial variations in mass and stiffness variation across the beam. The superelements have been widely applied for problems to reduce analysis and assembly time drastically for beam, plate, and box-beam problems (Ahmadian and Zangench 2002, Fan *et al*. 2004, Jiang and Olson 1993, Koko 1992, Vaziri 1996, Zivkovic *et al*. 2001, Nurse 2001, Qu and Selvam 2000, Cardona 2000, Tkachev 2000, Belyi 1993) and also permit the assembly of these substructures into a master structure. They often use a combinations of trigonometric and polynomial functions as interpolating functions (Ahmadian and Zangench 2002, Koko 1992). This is different from *p*-version FEM where only polynomial shape functions are used. Typically, upto five modes are required for dynamic analysis of rotating beams and accurate predictions upto the fifth frequency is therefore important and can require very high values of polynomial order. In *p*-version FEM, it is not practical to increase p (order of interpolating polynomial functions) to very high values. This is because high order polynomial functions are well known to be ill-conditioned (West *et al.* 1997), e.g., the computer can hardly find any difference between x^{10} and x^{11} with in $0 < x < 1$. This problem of *p*-version FEM can be addressed by using trigonometric functions along with polynomials which leads to the Fourier *p*-version of FEM (Leung and Chan 1998, Leung and Zhu 2004, Houmat 2001, Houmat 2001, Yongqiang 2006). It is found in these works that higher modes converge much faster using combinations of trigonometric functions and polynomials than when using polynomials alone. The Fourier-papproach can therefore be used to develop a superelement for a rotating beam. To the best of the author's knowledge, such superelements using mixed polynomial-trigonometric functions have not been developed for rotating beams, though they appear to be very attractive for this application.

In this paper, a superelement is developed using a combination of polynomials and Fourier series as shape functions, which results in an efficient formulation which can be used for practical vibration analysis and control problems for rotating structures such as helicopter rotor blades. A linear mass and quartic stiffness distribution which can represent rotor blades is included as a part of the FEM formulation. Superelements for the blades also permit easy integration with finite element models of the rotor hub and fuselage for helicopters and the rotor hub and tower for wind turbines.

2. Governing equation of rotating beams

A schematic of a rotating tapered beam is shown in Fig. 1. Here $m(x)$ and $EI(x)$ are the mass and flexural stiffness per unit length at a distance x from the axis of rotation, Ω is the rotational speed, $w(x, t)$ and $f(x, t)$ are the displacement in the Z direction and force per unit length, respectively. $T(x)$ is the centrifugal tensile load at a distance x from the axis of rotation, F is an axial force applied at the end of beam, L is the length of the beam and R is the distance of the beam root from the axis of rotation. In the present analysis, R is assumed to be zero. Such beams are good models for long slender structures such as helicopter rotor blades and wind turbine rotor blades whose cross-section dimensions are much smaller than the length (Kumar *et al.* 2007, Pawar and Ganguli 2006, Pawar and Ganguli 2005, Ganguli 2001). In the present work, we consider Euler-Bernoulli beams for the analysis of out-of-plane bending (flapping) vibration.

The governing partial differential equation with variable coefficients for out-of-plane (transverse) bending vibration of an Euler-Bernoulli rotating beam is given by Wang and Wereley (2004)

$$(EI(x)w'')'' + m(x)\ddot{w} - (T(x)w')' = f(x, t) \tag{1}$$

where

$$T(x) = \int_x^L m(x)\Omega^2(R + x)dx + F \tag{2}$$

where w' and w'' are first and second derivatives of w with respect to x, respectively.

Unfortunately, although the basic differential equation is linear, analytical solutions do not exist even for span wise constant properties.

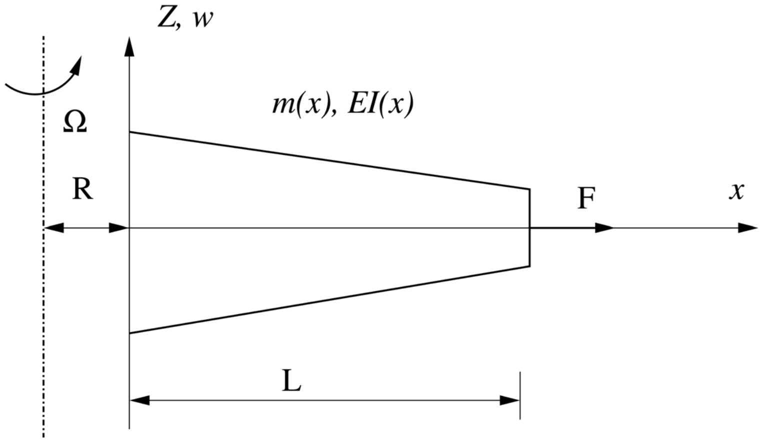

Fig. 1 Rotating tapered beam element geometry

3. Shape functions

Consider the transverse bending (flapping) vibrations of rotating beams. For bending elements, the shape functions are required to give displacement and slope continuity at the element interfaces. When considering the bending of a beam of unit length, the appropriate shape functions which are obtained by considering the quintic polynomial are given by

$$N(1,1) = 1 - \frac{331}{9}\xi^2 + \frac{326}{3}\xi^3 - 112\xi^4 + \frac{352}{9}\xi^5 \tag{3}$$

$$N(2,1) = \xi L\left(1 - \frac{22}{3}\xi + 17\xi^2 - 16\xi^3 + \frac{16}{3}\xi^4\right) \tag{4}$$

$$N(3,1) = \frac{128}{3}\xi^2 - \frac{1280}{9}\xi^3 + \frac{1408}{9}\xi^4 - \frac{512}{9}\xi^5 \tag{5}$$

$$N(4,1) = \frac{-128}{9}\xi^2 + \frac{256}{3}\xi^3 - 128\xi^4 + \frac{512}{9}\xi^5 \tag{6}$$

$$N(5,1) = \frac{25}{3}\xi^2 - \frac{466}{9}\xi^3 + \frac{752}{9}\xi^4 - \frac{352}{9}\xi^5 \tag{7}$$

$$N(6,1) = \xi L\left(-\xi + \frac{19}{3}\xi^2 - \frac{32}{3}\xi^3 + \frac{16}{3}\xi^4\right) \tag{8}$$

Here $\xi = x/L$ is the non-dimensional length of beam element.

For C^1 continuity requirement, the Fourier version is either $1 - \cos j\pi\xi$ or $(\xi - \xi^2)\sin j\pi\xi$ (Leung and Zhu 2004). Both functions and their first derivatives vanish at $\xi = 0$ and $\xi = 1$. For convenience, the first function is called as the cosine version and the second function as the sine version. Though the cosine version is simpler, it produces zero shear forces at the nodes and is too flexible for shear connections. The cosine version is not recommended when the structure has just one element. Therefore, the sine version is used in this analysis. The enhanced shape functions are given by

$$N(1, 1) = 1 - \frac{331}{9}\xi^2 + \frac{326}{3}\xi^3 - 112\xi^4 + \frac{352}{9}\xi^5 \tag{9}$$

$$N(2, 1) = \xi L\left(1 - \frac{22}{3}\xi + 17\xi^2 - 16\xi^3 + \frac{16}{3}\xi^4\right) \tag{10}$$

$$N(3, 1) = \frac{128}{3}\xi^2 - \frac{1280}{9}\xi^3 + \frac{1408}{9}\xi^4 - \frac{512}{9}\xi^5 \tag{11}$$

$$N(4, 1) = \frac{-128}{9}\xi^2 + \frac{256}{3}\xi^3 - 128\xi^4 + \frac{512}{9}\xi^5 \tag{12}$$

$$N(j + 4, 1) = (\xi - \xi^2)\sin j\pi\xi \tag{13}$$

$$N(k + 5, 1) = \frac{25}{3}\xi^2 - \frac{466}{9}\xi^3 + \frac{752}{9}\xi^4 - \frac{352}{9}\xi^5 \tag{14}$$

$$N(k + 6, 1) = \xi L\left(-\xi + \frac{19}{3}\xi^2 - \frac{32}{3}\xi^3 + \frac{16}{3}\xi^4\right) \tag{15}$$

where k indicates the number of sine terms used in the approximating polynomial function and the value of j varies from 1 to k.

The total number of degrees of freedom in this method is equal to that of the CFEM model plus the number of sine terms (k). The CFEM has two degrees of freedom at each node, namely, the deflection and rotation (slope). The sine terms correspond to the internal degrees of freedom of the element while the quintic polynomials correspond to the nodal degrees of freedom and two internal degrees of freedom at $L/4$ and $3L/4$. The number of sine terms can be increased for getting the solution to converge without increasing the number of elements. Thus, the mixed trigonometric and polynomial shape functions can be used to obtain convergence with one element only.

4. Superelement matrices

The kinetic energy for a rotating beam is given by

$$\mathcal{T} = \frac{1}{2}\int_0^L m(x)[\dot{w}(x, t)]^2 dx \tag{16}$$

where $\dot{w}(x, t)$ is derivative of $w(x, t)$ with respect to time t. The potential/strain energy is given by

$$\mathcal{U} = \frac{1}{2}\int_0^L EI(x)[w''(x, t)]^2 dx + \frac{1}{2}\int_0^L T(x)[w'(x, t)]^2 dx \tag{17}$$

where $T(x)$ is defined in Eq. (2).

The mass and stiffness matrices (**M** and **K**) for such a beam element can be obtained from the above energy expressions. The calculations for these matrices involve solving the following integrals

$$\mathbf{M} = \int_0^L m(x)\mathbf{N}\mathbf{N}^T dx \tag{18}$$

$$\mathbf{K} = \int_0^L EI(x)\mathbf{N}''(\mathbf{N}'')^T dx + \int_0^L T(x)\mathbf{N}'(\mathbf{N}')^T dx \tag{19}$$

Here superscript T in Eqs. (18) and (19) denotes the transpose of the matrix and **N** are the shape functions. Since $m(x)$, $EI(x)$ and $T(x)$ are inside the integrals in Eq. (18) and (19), the superelement mass and stiffness matrices can be evaluated easily for a given mass and stiffness distribution. Also, the element length is equal to the beam length. Note that for conventional FEM, the element stiffness matrix needs to be calculated at each element because of the centrifugal term, leads to considerable analysis time. The natural frequencies are obtained by solving the eigen value problem.

$$\mathbf{K}\Phi = \omega^2\mathbf{M}\Phi$$

5. Numerical results

The natural frequencies are calculated using this model for different rotational speeds using a single element with 10 Fourier (sine) terms. The results obtained from the present formulation are compared with those available in the literature.

5.1 Uniform beam

Table 1 shows an interesting comparison of non-dimensional natural frequencies of a uniform cantilever beam with results from Hodges and Rutkowsky (1981), Wright *et al.* (1982) and Wang and Wereley (2004). Two values of non-dimensional rotation speed, λ, are chosen for comparison. Here $\lambda^2 = \dfrac{\Omega^2}{EI_o/m_oL^4}$ where EI_o and m_o are the reference flexural stiffness and mass per unit length,

Table 1 Comparison of Non-dimensional natural frequencies of cantilever uniform beam

Mode	Present	Wang and Wereley (2004)	Wright *et al.* (1982)	Hodges *et al.* (1981)
		$\lambda = 0$		
1	3.5160	3.5160	3.5160	3.5160
2	22.0345	22.0345	22.0345	22.0345
3	61.6972	61.6972	61.6972	61.6972
4	120.902	120.902	120.902	N/A
5	199.860	199.860	199.860	N/A
		$\lambda = 12$		
1	13.1702	13.1702	13.1702	13.1702
2	37.6031	37.6031	37.6031	37.6031
3	79.6145	79.6145	79.6145	79.6145
4	140.534	140.534	140.534	N/A
5	220.537	220.536	220.536	N/A

Table 2 Comparison of Non-dimensional natural frequencies of hinged uniform beam

Mode	Present	Wang and Wereley (2004)	Wright *et al.* (1982)
		$\lambda = 0$	
1	0.0000	0.0000	0.0000
2	15.4182	15.4182	15.4182
3	49.9649	49.9649	49.9649
4	104.248	104.248	104.248
5	178.270	178.270	178.270
		$\lambda = 12$	
1	12.0000	12.0000	12.0000
2	33.7603	33.7603	33.7603
3	70.8373	70.8373	70.8373
4	126.431	126.431	126.431
5	201.123	201.122	201.122

respectively. All the frequencies predicted by superelement agree very well with the published results.

In this study, whole beam is considered as a single element and 10 Fourier (sine) terms are used with quintic polynomials as shape functions. Thus, the total number of degrees of freedom is 14 after application of cantilever boundary conditions and it gives fairly good accuracy for frequencies of up to the fifth mode. It is found that the same accuracy level for up to fifth mode frequency using *h*-FEM requires at least 50 finite elements, i.e., 102 degrees of freedom (2 degrees of freedom for every node) or 100 degree of freedom after applying cantilever boundary conditions.

Similar results are obtained for a hinged uniform beam and compared with the results from published literature in Table 2. These results also show excellent agreement.

5.2 Tapered rotating beam

For a better approximation to the practical rotor blade, we analyze it as a tapered beam. Although any type of tapered beam can be analyzed using the present approach, for illustrative purposes two different types of linearly tapered cantilever beams are selected from the published literature by Hodges and Rutkowsky (1981) and Wright *et al.* (1982).

The beam element used in this analysis is a tapered element. The element stiffness and mass matrices are calculated exactly for a tapered element. Thus the tapered beam is not idealized using several piece-wise uniform beam elements as often done in conventional formulation. Therefore, this approach gives more accurate results with one element.

In general, we assume that variation of mass along the beam length is defined as

$$m(x) = m_0(1 - \alpha\xi) \tag{20}$$

where m_0 corresponds to the value of mass per unit length at the thick end of the beam ($\xi = 0$), α is the taper parameter such that $0 < \alpha < 1$. $\alpha \neq 1$, which results in a singularity at $\xi = 1$. flexural stiffness variation along the length of beam element is defined as

$$EI(x) = EI_0(1 - \beta_1\xi - \beta_2\xi^2 - \beta_3\xi^3 - \beta_4\xi^4) \tag{21}$$

where EI_0 correspond to the value of flexural rigidity at the thick end of the beam ($\xi = 0$). Here β_i, $i = 1$ to 4 are taper parameters for stiffness distribution. These parameters can be determined by α for beams with a rectangular cross-sectional area and thickness varying along the beam length. However, as with the example studied by Wright *et al*. (1982), the taper parameters for mass and flexural stiffness are not necessarily related. They are independent variables. However, these parameters should not result in a singularity for flexural stiffness at $\xi = 1$.

5.2.1 Example 1 (Linear mass, cubic stiffness, cantilevered beam)

In this example, the taper is such that the variations of the mass per unit length $m(x)$, and the bending flexural rigidity $EI(x)$ at a distance x from the thick end are governed by the following expressions

$$m(x) = m_0(1 - 0.5\xi) \tag{22}$$

and

$$EI(x) = EI_0(1 - 0.5\xi)^3 \tag{23}$$

Taper parameters are

$$\alpha = 0.5 \qquad \beta_1 = 3\alpha = 1.5$$
$$\beta_2 = -3\alpha^2 = -0.75, \qquad \beta_3 = \alpha^3 = 0.125 \qquad \beta_4 = 0$$

Table 3 Comparison of Non-dimensional natural frequencies of tapered cantilever beam under different rotational speeds (Example 1)

	First mode			Second mode			Third mode		
λ	Present	Wang and Wereley (2004)	Hodges and Rutkowsky (1981)	Present	Wang and Wereley (2004)	Hodges and Rutkowsky (1981)	Present	Wang and Wereley (2004)	Hodges and Rutkowsky (1981)
0	3.8238	3.8238	3.8238	18.3173	18.3173	18.3173	47.2649	47.2648	47.2648
1	3.9866	3.9866	3.9866	18.4740	18.4740	18.4740	47.4173	47.4173	47.4173
2	4.4368	4.4368	4.4368	18.9366	18.9366	18.9366	47.8717	47.8716	47.8716
3	5.0927	5.0927	5.0927	19.6839	19.6839	19.6839	48.6190	48.6190	48.6190
4	5.8788	5.8788	5.8788	20.6852	20.6852	20.6852	49.6457	49.6456	49.6456
5	6.7434	6.7434	6.7345	21.9053	21.9053	21.9053	50.9338	50.9338	50.9338
6	7.6551	7.6551	7.6551	23.3093	23.3093	23.3093	52.4633	52.4633	52.4633
7	8.5956	8.5956	8.5956	24.8647	24.8647	24.8647	54.2125	54.2124	54.2124
8	9.5540	9.5540	9.5540	26.5437	26.5437	26.5437	56.1595	56.1595	56.1595
9	10.5239	10.5239	10.5239	28.3227	28.3227	28.3227	58.2834	58.2833	58.2833
10	11.5015	11.5015	11.5015	30.1827	30.1827	30.1827	60.5639	60.5639	60.5639
11	12.4845	12.4845	12.4845	32.1084	32.1085	32.1085	62.9829	62.9829	62.9829
12	13.4711	13.4711	13.4711	34.0877	34.0877	34.0877	65.5237	65.5237	65.5237

This type of tapered beam is used by Hodges and Rutkowsky (1981) for analysis. These equations cover all beams having a solid rectangular cross section with constant width and linearly varying depth. The results obtained for this case are compared with those obtained by Wang and Wereley (2004) and Hodges and Rutkowsky (1981) in Table 3. Since results for higher modes are not available in the published literature, a comparison of only three modes is shown in this table. The results obtained using superelement show excellent agreement with the published results.

Wang and Wereley (2004) used a single spectral finite element with the first 80 terms in the Frobenius power series for similar accuracy level while Hodges and Rutkowsky (1981) used a variable order finite element with 15th order polynomials. The present analysis uses a single superelement with only 10 sine terms as interpolating functions to get comparable results.

5.2.2 Example 2 (Linear mass, linear stiffness, cantilevered beam)

In the second example, the tapered beam used by Wright *et al.* (1982) is considered. For this particular problem, the taper is such that both the mass per unit length $m(x)$, and the bending flexural rigidity $EI(x)$ vary linearly along the length of the beam so that,

$$m(x) = m_0(1 - 0.8\xi) \tag{24}$$

and

$$EI(x) = EI_0(1 - 0.95\xi) \tag{25}$$

Taper parameters coresponding to Eqs. (20) and (21) stated earlier are

$$\alpha = 0.8, \qquad \beta_1 = 0.95, \qquad \beta_2 = \beta_3 = \beta_4 = 0$$

Table 4 Comparison of Non-dimensional natural frequencies of tapered cantilever beam under different rotational speeds (Example 2)

	First mode			Second mode			Third mode		
λ	Present	Wang and Wereley (2004)	Wright *et al.* (1982)	Present	Wang and Wereley (2004)	Wright *et al.* (1982)	Present	Wang and Wereley (2004)	Wright *et al.* (1982)
0	5.2738	5.2738	5.2738	24.0041	24.0041	24.0041	59.9702	59.9708	59.9701
1	5.3903	5.3903	5.3903	24.1069	24.1069	24.1069	60.0697	60.0703	60.0696
2	5.7249	5.7249	5.7249	24.4130	24.4129	24.4130	60.3670	60.3676	60.3669
3	6.2402	6.2402	6.2402	24.9149	24.9148	24.9149	60.8591	60.8598	60.8590
4	6.8928	6.8928	6.8928	25.6013	25.6013	25.6013	61.5469	61.5420	61.5412
5	7.6443	7.6443	7.6443	26.4581	26.4581	26.4581	62.4070	62.4078	62.4069
6	8.4653	8.4653	8.4653	27.4693	27.4692	27.4693	63.4484	63.4494	63.4483
7	9.3347	9.3347	9.3347	28.6185	28.6184	28.6185	64.6567	64.6579	64.6566
8	10.2379	10.2379	10.2379	29.8894	29.8893	29.8894	66.0223	66.0238	66.0222
9	11.1650	11.1651	11.1650	31.2669	31.2667	31.2669	67.5352	67.5370	67.5351
10	12.1092	12.1092	12.1092	32.7369	32.7367	32.7369	69.1852	69.1875	69.1851
11	13.0657	13.0657	13.0657	34.2871	34.2868	34.2871	70.9623	70.9653	70.9622
12	14.0313	14.0313	14.0313	35.9064	35.9060	35.9064	72.8566	72.8604	72.8565

Table 5 Comparison of Non-dimensional natural frequencies of tapered cantilever beam under different rotational speeds (Example 2)

	Fourth mode			Fifth mode		
λ	Present	Wang and Wereley (2004)	Wright *et al.* (1982)	Present	Wang and Wereley (2004)	Wright *et al.* (1982)
0	112.910	112.892	112.909	183.029	183.473	183.024
1	113.010	112.992	113.009	183.129	183.576	183.124
2	113.308	113.290	113.307	183.429	183.887	183.424
3	113.803	113.784	113.803	183.928	184.404	183.923
4	114.493	114.472	114.492	184.623	185.127	184.619
5	115.373	115.351	115.372	185.514	186.055	185.509
6	116.439	116.414	116.439	186.596	187.186	186.591
7	117.686	117.658	117.685	187.867	188.520	187.862
8	119.108	119.075	119.107	189.321	190.055	189.316
9	120.697	120.659	120.696	190.955	191.793	190.950
10	122.447	122.403	122.446	192.764	193.734	192.759
11	124.351	124.298	124.350	194.743	195.882	194.737
12	126.401	126.336	126.401	196.885	198.243	196.880

As mentioned by Wright *et al.* (1982), this beam design is used in wind turbine blades. As shown in the preceding equation, when $\xi = 1$, the flexural stiffness drops to 5% of the initial value of EI_0. The singularity is very close to $\xi = 1$, which results in a slower convergence of the results. The results obtained for this case are compared with those of Wright *et al.* (1982). Wang and Wereley 2004 [16] have also analyzed the same beam using a spectral finite element method. Table 4 shows the comparison of our results with the published works for the first three modes and Table 5 shows the comparison for fourth and fifth modes.

Wang and Wereley (2004) used a single spectral finite element with as many as 350 terms in Frobenius power series. The present analysis uses one superelement with only 10 sine terms as the interpolating functions and the results compare very well with published work.

5.2.3 Example 3 (Linear mass, linear stiffness, hinged beam)

In the third example, the tapered beam used by Wright *et al.* (1982) with hinged boundary conditions is considered. The same superelement model can be used to obtain the natural frequencies of hinged tapered beams with different geometric boundary conditions.

For this particular problem, the taper is such that both the mass per unit length $m(x)$, and the bending flexural rigidity $EI(x)$ vary linearly along the length of the beam so that,

$$m(x) = m_0(1 - 0.8\xi) \tag{26}$$

and

$$EI(x) = EI_0(1 - 0.95\xi) \tag{27}$$

Taper parameters are thus identical to those considered in Example 2 and the only difference is in

Table 6 Comparison of Non-dimensional natural frequencies of tapered hinged beam under different rotational speeds (Example 3)

	First mode			Second mode			Third mode		
λ	Present	Wang and Wereley (2004)	Wright *et al.* (1982)	Present	Wang and Wereley (2004)	Wright *et al.* (1982)	Present	Wang and Wereley (2004)	Wright *et al.* (1982)
0	0.0	0.0	0.0	16.7328	16.7328	16.7328	48.4692	48.4696	48.4691
1	1.0	1.0	1.0	16.8711	16.8711	16.8711	48.5872	48.5876	48.5872
2	2.0	2.0	2.0	17.2794	17.2793	17.2794	48.9395	48.9399	48.9395
3	3.0	3.0	3.0	17.9390	17.9389	17.9388	49.5210	49.5214	49.5210
4	4.0	4.0	4.0	18.8233	18.8233	18.8233	50.3234	50.3239	50.3234
5	5.0	5.0	5.0	19.9023	19.9022	19.9022	51.3360	51.3366	51.3360
6	6.0	6.0	6.0	21.1457	21.1456	21.1456	52.5464	52.5471	52.5463
7	7.0	7.0	7.0	22.5260	22.5259	22.5260	53.9406	53.9415	53.9405
8	8.0	8.0	8.0	24.0195	24.0194	24.0195	55.5043	55.5054	55.5042
9	9.0	9.0	9.0	25.6061	25.6059	25.6060	57.2231	57.2245	57.2230
10	10.0	10.0	10.0	27.2692	27.2690	27.2692	59.0828	59.0847	59.0828
11	11.0	11.0	11.0	28.9956	28.9953	28.9956	61.0701	61.0727	61.0700
12	12.0	12.0	12.0	30.7745	30.7741	30.7745	63.1723	63.1758	63.1722

Table 7 Comparison of Non-dimensional natural frequencies of tapered hinged beam under different rotational speeds (Example 3)

	Fourth mode			fifth mode		
λ	Present	Wang and Wereley (2004)	Wright *et al.* (1982)	Present	Wang and Wereley (2004)	Wright *et al.* (1982)
0	97.1709	97.1599	97.1704	163.004	163.277	163.002
1	97.2828	97.2717	97.2823	163.114	163.388	163.111
2	97.6177	97.6062	97.6172	163.441	163.723	163.438
3	98.1732	98.1611	98.1727	163.985	164.281	163.983
4	98.9455	98.9324	98.9450	164.744	165.059	164.741
5	99.9292	99.9147	99.9287	165.714	166.055	165.771
6	101.118	101.102	101.117	166.892	167.268	166.889
7	102.504	102.485	102.504	168.272	168.695	168.269
8	104.080	104.058	104.079	169.851	170.333	169.847
9	105.836	105.809	105.835	171.621	172.180	171.618
10	107.762	107.730	107.762	173.577	174.236	173.573
11	109.851	109.810	109.850	175.711	176.500	175.708
12	112.091	112.040	112.090	178.019	178.978	178.105

the boundary conditions. The taper parameters are given by

$$\alpha = 0.8, \qquad \beta_1 = 0.95, \qquad \beta_2 = \beta_3 = \beta_4 = 0$$

Earlier, Wright *et al*. (1982), and Wang and Wereley (2004) have discussed this type of tapered beam with hinged boundary conditions. Wang and Wereley (2004) have used one spectral finite element with 350 terms in Frobenius functions to obtain the results. The method used by them is based on a similar principle of using power series as that of Wright *et al.* (1982). The results obtained using the present approach for first three modes are compared with those of published literature in Table 6. Table 7 presents comparison of fourth and fifth modes of same beam. Again, excellent agreement is obtained with the published results.

5.3 Effect of hub radius and slenderness ratio

In this section, the effect of hub radius *R* (Fig. 1) and slenderness ratio L/L_0 is studied for a linear mass, cubic stiffness, tapered cantilever beam as presented in example 1. The first three natural frequencies are obtained by varying the hub radius as a percentage of the total length of the beam

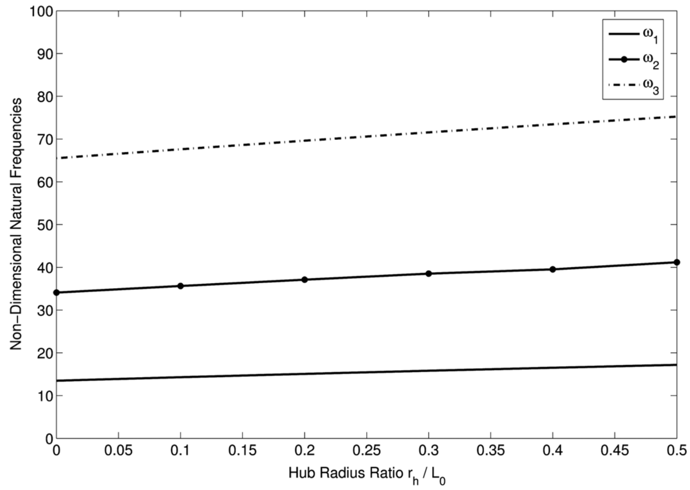

Fig. 2 Effect of hub radius on first three natural frequencies at $\lambda = 12$

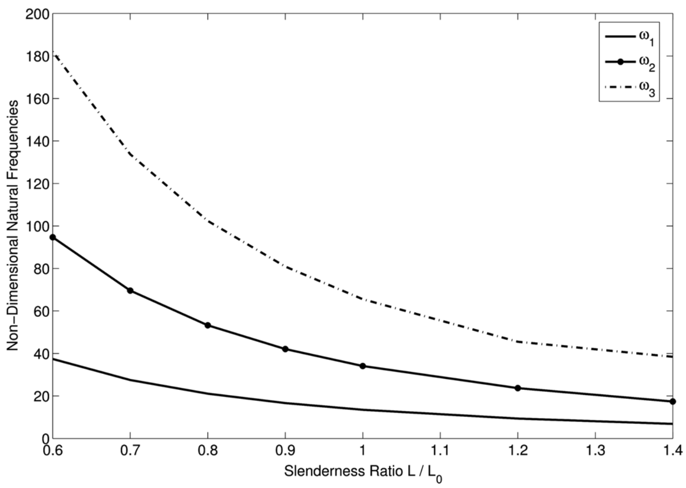

Fig. 3 Effect of slenderness ratio on first three natural frequencies at $\lambda = 12$

and the slenderness ratio at a constant non-dimensional rotation speed of $\lambda = 12$ and are presented in Figs. 2 and 3. Here L_0 is the reference length. The natural frequencies are non-dimensionalized with EI_0/mL_0^4. The non-dimensional natural frequencies increase with increase in hub radius because of increased centrifugal stiffening of the beam and the natural frequencies decrease with increase in slenderness ratio.

6. Conclusions

A superelement is used for finding the natural frequencies of rotating uniform and tapered beams, with cantilever and hinged boundary conditions. The shape functions used for modeling the finite element consist of a combination of product of polynomial functions and Fourier series, in addition to quintic polynomials. The classical *h*-FEM is therefore enhanced using the trigonometric functions to form the superelement. Since Fourier series are well behaved, the limitation of the higher order polynomial functions of being ill-conditioned, is removed. The results obtained from the current approach show an excellent match with the results obtained from different methods in the published literature for uniform and tapered rotating beams with cantilever and hinged boundary conditions. The superelement is easy to use for non-uniform mass and stiffness distributions which occur in helicopter and wind turbine blades. The stiffness matrix of even uniform rotating beams vary due to centrifugal effects and need to be calculated for each element in the conventional FEM formulation, leads to considerable analysis and assembly time, which is saved by the superelement.

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