CONTRACTIVE AND COMPLETELY CONTRACTIVE HOMOMORPHISMS OF PLANAR ALGEBRAS

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ABSTRACT. We consider contractive homomorphisms of a planar algebra $A(\Omega)$ over a finitely connected bounded domain $\Omega \subseteq \mathbb{C}$ and ask if they are necessarily completely contractive. We show that a homomorphism $\rho : A(\Omega) \to B(H)$ for which $\dim(A(\Omega)/\ker \rho) = 2$ is the direct integral of homomorphisms $\rho_T$ induced by operators on two-dimensional Hilbert spaces via a suitable functional calculus $\rho_T : f \mapsto f(T)$, $f \in A(\Omega)$. It is well known that contractive homomorphisms $\rho_T$ induced by a linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ are necessarily completely contractive. Consequently, using Arveson’s dilation theorem for completely contractive homomorphisms, one concludes that such a homomorphism $\rho_T$ possesses a dilation. In this paper, we construct this dilation explicitly. In view of recent examples discovered by Dritschel and McCullough, we know that not all contractive homomorphisms $\rho_T$ are completely contractive even if $T$ is a linear transformation on a finite-dimensional Hilbert space. We show that one may be able to produce an example of a contractive homomorphism $\rho_T$ of $A(\Omega)$ which is not completely contractive if an operator space which is naturally associated with the problem is not the MAX space. Finally, within a certain special class of contractive homomorphisms $\rho_T$ of the planar algebra $A(\Omega)$, we construct a dilation.

1. Introduction

All our Hilbert spaces are over complex numbers and are assumed to be separable. Let $T \in B(H)$, the algebra of bounded operators on $H$. The operator $T$ induces a homomorphism $\rho_T : p \mapsto p(T)$, where $p$ is a polynomial. Equip the polynomial ring with the supremum norm on the unit disc, that is, $\|p\| = \sup \{ |p(z)| : z \in \mathbb{D} \}$. A well-known inequality due to von Neumann (cf. [18]) asserts that $\rho_T$ is contractive, that is, $\|\rho_T\| \leq 1$, if and only if the operator

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$T$ is a contraction. Thus in this case, contractivity of the homomorphism $\rho_T$ is equivalent to the operator $T$ being a contraction. As is well known, Sz.-Nagy [24] showed that a contraction $T$ on a Hilbert space $H$ dilates to a unitary operator $U$ on a Hilbert space $K$ containing $H$, that is, $Pp(U)h = p(T)h$ for all $h \in H$ and any polynomial $p$, where $P : K \to H$ is the projection of $K$ onto $H$. The unitary operator $U$ has a continuous functional calculus and hence induces a $*$-homomorphism $\varphi_U : C(\sigma(U)) \to B(K)$. It is easy to check that $P[(\varphi_U)(A(\Omega))(f)]h = \rho_T(f)$, for $f$ in $A(\Omega)$, where $A(\Omega)$ is the closure of the polynomials with respect to the supremum norm on the disc $\Omega$.

Let $\Omega$ be a finitely connected bounded domain in $\mathbb{C}$. We make the standing assumption that the boundary of $\Omega$ is the disjoint union of simple analytic closed curves. Let $T$ be a bounded linear operator on the Hilbert space $H$ with spectrum $\sigma(T) \subseteq \Omega$. Given a rational function $r = p/q$ with no poles in the spectrum $\sigma(T)$, there is the natural functional calculus $r(T) = p(T)q(T)^{-1}$. Thus $T$ induces a unital homomorphism $\rho_T = r(T)$ on the algebra of rational functions $\text{Rat}(\Omega)$ with poles off $\Omega$. Let $A(\Omega)$ be the closure of $\text{Rat}(\Omega)$ with respect to the norm $\|r\| := \sup\{|r(z)| : z \in \Omega\}$. Since functions holomorphic in a neighborhood of $\bar{\Omega}$ can be approximated by rational functions with poles off $\bar{\Omega}$, it follows that they belong to $A(\Omega)$.

The homomorphism $\rho_T$ is said to be dilatable if there exists a normal operator $N$ on a Hilbert space $K \supseteq H$ with $\sigma(N) \subseteq \partial \bar{\Omega}$ such that the induced homomorphism $\varphi_N : C(\sigma(N)) \to B(K)$, via the functional calculus for the normal operator $N$, satisfies the relation

$P[(\varphi_N)(A(\Omega))(f)]h = \rho_T(f)h$,

for $h$ in $H$ and $f$ in $A(\Omega)$. Here $P : K \to H$ is the projection of $K$ onto $H$.

The observations about the disk prompt two basic questions:

(i) When is $\rho_T$ contractive?
(ii) Do contractive homomorphisms $\rho_T$ necessarily dilate?

For the disc algebra, the answer to the first question is given by von Neumann’s inequality while the answer to the second question is affirmative by Sz.-Nagy’s dilation theorem. If the domain $\Omega$ is simply connected these questions can be reduced to that of the disc (cf. [23]).

If the domain $\Omega$ is the annulus, while no satisfactory answer to the first question is known, the answer to the second question was shown to be affirmative by Agler (cf. [4]).

If $\rho_T : A(\Omega) \to M_2$ is a homomorphism induced by an operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ then it is possible to obtain a characterization of contractivity and then use it to show that the second question has an affirmative answer. We do this in Section 3.2. In Section 2, we show that a larger class of contractive homomorphisms, we call them contractive homomorphisms of rank 2, dilate. This is done by proving that the rank 2 homomorphisms are direct integrals of homomorphisms induced by two-dimensional operators.
Arveson (cf. [5] and [6]) has shown that the existence of a dilation of a contractive homomorphism \( \rho \) of the algebra \( \mathcal{A}(\Omega) \) is equivalent to complete contractivity of the homomorphism \( \rho \). We recall some of these notions in greater detail in Section 4. We then show how one may proceed to possibly construct an example of a contractive homomorphism of the algebra \( \mathcal{A}(\Omega) \) which does not dilate.

In the final section of the paper, we obtain a general criterion for contrac-
tivity. This involves a factorization of a certain positive definite kernel. More importantly, we outline a scheme for constructing the dilation of a homomor-
phism \( \rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_n \) induced by an operator \( T \) with distinct eigenvalues. This scheme is a generalization of the construction of the dilation in Section 3.2.

2. Homomorphisms of rank two

A homomorphism \( \rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H}) \) is said to be of rank \( n \) if it has the property \( \dim (\mathcal{A}(\Omega)/\ker \rho) = n \). In this section, we shall begin construc-
tion of dilation for homomorphisms of rank 2. Nakazi and Takahashi showed that contractive homomorphisms \( \rho : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H}) \) of rank 2 are completely contractive for any uniform sub-algebra of the algebra of continuous func-
tions \( C(\overline{\Omega}) \) (see [17]). We would like to mention here that a generalization of this result was obtained by Meyer in Theorem 4.1 of [12]. He showed that given a commutative unital closed subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{K}) \) (for some Hilbert
space \( \mathcal{K} \)) and a positive integer \( d \), any \( d-1 \) contractive unital homomorphism \( \rho : \mathcal{A} \to \mathcal{M}_d \) is completely contractive. In what follows, we construct explicit dilations for homomorphisms from \( \mathcal{A}(\Omega) \) to \( \mathcal{B}(\mathcal{H}) \) of rank two.

We first show that any homomorphism \( \rho \) of rank 2 is the direct integral of homomorphisms of the form \( \rho_T \) as defined in the introduction, where \( T \in \mathcal{M}_2 \). The existence of dilation of a contractive homomorphism \( \rho_T \) induced by a two-dimensional operator \( T \) is established in [13] by showing that the homomorphism \( \rho_T \) must be completely contractive. It then follows that every contractive homomorphism \( \rho \) of rank 2 must be completely contractive. This implies by Arveson’s theorem that they possess a dilation. However, it is not always easy to construct the dilation whose existence is guaranteed by the theorem of Arveson. In this case, we shall explicitly construct the dilation of a homomorphism of rank 2. This is achieved by constructing the dilation of a contractive homomorphism of the form \( \rho_T \) for a two-dimensional operator \( T \).

**Lemma 1.** If \( \rho_T : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{L}) \) is a homomorphism of rank two, then up to unitary equivalence, the Hilbert space \( \mathcal{L} \) is a direct integral

\[
\mathcal{L} = \bigoplus_{\lambda} \mathcal{L}_\lambda d\nu(\lambda),
\]
where each \( L_\lambda \) is two-dimensional. In this decomposition, the operator \( T \) is of the form
\[
T = \int_\Lambda^{\oplus} \begin{pmatrix} z_1(\lambda) & c(\lambda) \\ 0 & z_2(\lambda) \end{pmatrix} d\nu(\lambda).
\]

Proof. To begin with, it is easy to see (see Lemma 1 of [17]) that \( L \) is a direct sum of two Hilbert spaces \( H \) and \( K \) and the operator \( T : H^{\oplus} K \to H^{\oplus} K \) is of the form
\[
\begin{pmatrix} z_1 I_H & C \\ 0 & z_2 I_K \end{pmatrix}, \text{ with } z_1, z_2 \in \Omega \text{ or } \begin{pmatrix} z I_H & C \\ 0 & z I_K \end{pmatrix}, \text{ with } z \in \Omega,
\]
where \( C \) is a bounded operator from \( K \) to \( H \). Now if we put \( K_0 = (\ker C)^\perp \), \( K_1 = \ker C \), \( H_0 = \text{Ran } C \) and \( H_1 = (\text{Ran } C)^\perp \), then with respect to the decomposition \( K = K_0 \oplus K_1 \) and \( H = H_0 \oplus H_1 \) we have
\[
C = \begin{pmatrix} \hat{C} & 0 \\ 0 & 0 \end{pmatrix}
\]
where the operator \( \hat{C} \) is from \( K_0 \) to \( H_0 \). The polar decomposition of \( \hat{C} \) then yields \( \hat{C} = VP \), where the operator \( V \) is unitary and \( P \) is positive. We apply the spectral theorem to the positive operator \( P \) and conclude that there exists a unitary operator \( \Gamma : \int_\Lambda^{\oplus} H_\lambda d\nu(\lambda) \to K_0 \) which intertwines the multiplication operator \( M \) on the Hilbert space \( \int_\Lambda^{\oplus} H_\lambda d\nu(\lambda) \) and \( P \).

Now notice that the operator \( T : H^{\oplus} K \to H^{\oplus} K \) can be rewritten as
\[
\begin{pmatrix} z_1 I_{H_1} & 0 & \hat{C}_{K_0 \to H_0} & 0 \\ 0 & z_1 I_{H_0} & 0 & 0 \\ 0 & 0 & z_2 I_{K_0} & 0 \\ 0 & 0 & 0 & z_2 I_{K_1} \end{pmatrix}.
\]
Interchanging the third and the second column and then the second and third row, which can be effected by a unitary operator, we see that the operator \( T \) is unitarily equivalent to the direct sum of a diagonal operator \( D \) and an operator \( \hat{T} \) of the form
\[
\begin{pmatrix} z_1 I_{H_0} & \hat{C}_{K_0 \to H_0} \\ 0 & z_2 I_{K_0} \end{pmatrix},
\]
where \( \hat{C} \) has dense range. It is clear that if we conjugate the operator \( \hat{T} \) by the operator \( I_{H_0} \oplus U_{K_0 \to K_0} \), where \( U \) is any unitary operator identifying \( H_0 \) and \( K_0 \), then we obtain a unitarily equivalent copy of \( \hat{T} \) (again, denoted by \( \hat{T} \)), which is of the form
\[
\begin{pmatrix} z_1 I_{H_0} & \hat{C}_{K_0 \to H_0} U_{H_0 \to K_0} \\ 0 & z_2 I_{H_0} \end{pmatrix}.
\]
Now, if we apply the polar decomposition to \( \hat{C} \) then we see that the off diagonal entry is a positive operator on \( H_0 \). One then sees that \( \hat{T} \) is unitarily
equivalent to
\[
\begin{pmatrix}
  z_1 I^\oplus_{\mathcal{H}_\lambda d\nu(\lambda)} & M \\
  z_2 I^\oplus_{\mathcal{H}_\lambda d\nu(\lambda)} & 0
\end{pmatrix}
\]
via conjugation using the operator \( \Gamma \oplus \Gamma \). We need to conjugate this operator one more time using the unitary \( W \) that identifies \( \int_A^{\oplus} \mathcal{H}_\lambda d\nu(\lambda) \oplus \int_A^{\oplus} \mathcal{H}_\lambda d\nu(\lambda) \) and \( \int_A^{\oplus} \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda d\nu(\lambda) \), where \( W(s_1 \oplus s_2)(\lambda) = s_1(\lambda) \oplus s_2(\lambda) \) for \( s_1 \oplus s_2 \in \int_A^{\oplus} \mathcal{H}_\lambda d\nu(\lambda) \). It is easy to calculate \( WTW^* \) and verify the claim. \( \square \)

In view of the above lemma, it is now enough to consider dilations of homomorphisms \( \rho_T \) where \( T \) is a linear transformation on \( \mathbb{C}^2 \).

3. Dilations and Abrahamse-Nevanlinna-Pick interpolation

3.1. Consider any reproducing kernel Hilbert space \( \mathcal{H}_K \) of holomorphic functions on \( \Omega \) with \( K : \Omega \times \Omega \to \mathbb{C} \) as the kernel. Assume that the multiplication operator \( M \) by the independent variable \( z \) is bounded. Then \( M^*(K(\cdot, z)) = \bar{z} K(\cdot, z) \) and it is clear by differentiation that \( M^* \partial_z K(\cdot, z) = K(\cdot, z) + \bar{z} \partial_z K(\cdot, z) \).

The matrix representation of the operator \( M^* \) restricted to the subspace \( \mathcal{M} \) spanned by the two vectors \( K(\cdot, z_1) \) and \( K(\cdot, z_2) \) has two distinct eigenvalues \( \bar{z}_1 \) and \( \bar{z}_2 \). Similarly, the operator \( M^* \) restricted to the subspace \( \mathcal{N} \) spanned by the two vectors \( K(\cdot, z) \) and \( \partial_z K(\cdot, z) \) has only one eigenvalue \( \bar{z} \) of multiplicity 2. In the lemma below, we identify certain 2-dimensional subspaces of \( \mathcal{H}_K \) which are invariant under the multiplication operator \( M^* \) and then find out the form of the matrix. The reproducing kernel \( K \) satisfies

\[
\begin{align*}
(3.1a) \quad & K(z_1, z_2) = \langle K(\cdot, z_2), K(\cdot, z_1) \rangle, \quad z_1, z_2 \in \Omega, \\
(3.1b) \quad & \langle \partial_z K(\cdot, u), \partial_z K(\cdot, z) \rangle = \langle K(\cdot, u), \partial_z K(\cdot, z) \rangle, \quad u, z \in \Omega.
\end{align*}
\]

Using (3.1) and applying the Gram-Schmidt orthogonalization process to the set \( \{ K(\cdot, z_1), K(\cdot, z_2) \} \), we get the orthonormal pair of vectors

\[
e(z_1) = \frac{K(\cdot, z_1)}{K(z_1, z_1)^{1/2}}
\]

and

\[
f(z_1, z_2) = \frac{K(z_1, z_1)K(\cdot, z_2) - K(z_2, z_2)K(\cdot, z_1)}{K(z_1, z_1)^{1/2}(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}}.
\]

Now for any \( \mu \in \mathbb{D} \), the pair of vectors

\[
h_1(z_1, z_2) = \begin{pmatrix} 0 \\ e(z_1) \end{pmatrix} \quad \text{and} \quad h_2(z_1, z_2) = \begin{pmatrix} (1 - |\mu|^2)^{1/2}e(z_2) \\ \mu f(z_1, z_2) \end{pmatrix}
\]
are orthonormal in $\mathcal{H}_K \oplus \mathcal{H}_K$. Similarly, orthonormalizing the pair of vectors \{\(K(\cdot, z), \bar{\partial}_z K(\cdot, z)\)}, using (3.1b), we see that the pair \(\{e(z), f(z)\}\), where
\[
e(z) = \frac{K(\cdot, z)}{K(z, z)^{1/2}}
\]
and
\[
f(z) = \frac{K(z, z)\bar{\partial}_z K(\cdot, z) - \langle \bar{\partial}_z K(\cdot, z), K(\cdot, z) \rangle K(\cdot, z)}{K(z, z)^{1/2} \left( K(z, z) \|\bar{\partial}_z K(\cdot, z)\|^2 - \|\langle \bar{\partial}_z K(\cdot, z), K(\cdot, z) \rangle\|^2 \right)^{1/2}}
\]
are orthonormal. Now, for any \(\lambda \in \overline{\mathbb{D}}\),
\[
k_1(z) = \begin{pmatrix} 0 \\ e(z) \end{pmatrix} \text{ and } k_2(z) = \begin{pmatrix} (1 - |\lambda|^2)^{1/2} e(z) \\ \lambda f(z) \end{pmatrix}
\]
form a set of two orthonormal vectors in $\mathcal{H}_K \oplus \mathcal{H}_K$.

Note that from the definition of $M^*$ it follows that $M^*e(z_1) = \bar{z}_1 e(z_1)$ for all $z_1 \in \Omega$. Therefore we have $(M^* \oplus M^*)h_1(z_1, z_2) = \bar{z}_1 h_1(z_1, z_2)$. Now,
\[
(M^* \oplus M^*)f(z_1, z_2) = \frac{K(z_1, z_1)\bar{z}_2 K_{\alpha}(\cdot, z_2) - \bar{K}(z_2, z_2)\bar{z}_1 K(\cdot, z_1)}{K(z_1, z_1)^{1/2}(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}}
\]
\[
= \bar{z}_2 f(z_1, z_2) + \frac{(\bar{z}_2 - \bar{z}_1)K(z_1, z_2)}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} e(z_1).
\]

It follows that $M$ is invariant under $M^* \oplus M^*$. In particular, we have
\[
(M^* \oplus M^*)h_2(z_1, z_2) = \begin{pmatrix} (1 - |\mu|^2)^{1/2} M^* e(z_2) \\ \mu M^* f(z_1, z_2) \end{pmatrix}
\]
\[
= \begin{pmatrix} (1 - |\mu|^2)^{1/2} \bar{z}_2 e(z_2) \\ \lambda (\bar{z}_2 - \bar{z}_1)K(z_1, z_2) / (K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2} e(z_1) \end{pmatrix}
\]
\[
= \bar{z}_2 \left( (1 - |\mu|^2)^{1/2} e(z_2) \right) + \begin{pmatrix} 0 \\ \mu f(z_1, z_2) \end{pmatrix}
\]
\[
= \bar{z}_2 h_2(z_1, z_2) + \mu \frac{(\bar{z}_2 - \bar{z}_1)K(z_1, z_2)}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} h_1(z_1, z_2),
\]
where we have absorbed the argument of $K(z_1, z_2)$ in $\mu$.

Now recall that $(M^* - \bar{z})K(\cdot, z) = 0$. Differentiating with respect to $\bar{z}$, we obtain $M^* \bar{\partial}_{\bar{z}} K(\cdot, z) = K(\cdot, z) + \bar{z} \bar{\partial}_z K(\cdot, z)$. Thus the subspace $\mathcal{N}$ spanned by the vectors $k_1(z), k_2(z)$ is invariant under $M^*$. A little more computation, similar to the one above, gives us the matrix representation of the restriction of the operator $M^* \oplus M^*$ to the subspace $\mathcal{N}$.

So, we have proved the following lemma.

**Lemma 2.** The two-dimensional space $M$ spanned by the vectors $h_1(z_1, z_2)$ and $h_2(z_1, z_2)$ is an invariant subspace for the operator $M^* \oplus M^*$ on $\mathcal{H}_K \oplus$
\( \mathcal{H}_K \) and the restriction of this operator to the subspace \( \mathcal{M} \) has the matrix representation

\[
\begin{pmatrix}
\hat{z}_1 & \frac{\mu(\bar{z}_2 - \bar{z}_1) |K(z_1, z_2)|}{(K(z_1, z_1) K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} \\
0 & \hat{z}_2
\end{pmatrix}.
\]

Similarly, the two-dimensional space \( \mathcal{N} \) spanned by the two vectors \( k_1(z) \), \( k_2(z) \) is an invariant subspace for the operator \( M^* \oplus M^* \) on \( \mathcal{H} \oplus \mathcal{H} \) and the restriction of this operator to the subspace \( \mathcal{N} \) has the matrix representation

\[
\begin{pmatrix}
\hat{z} & \frac{\lambda K(z, z)}{\|K(z, z)\| - |\partial_2 K(z, z)|^2} \\
0 & \hat{\bar{z}}
\end{pmatrix}.
\]

Let \( \mu, \lambda \) be a pair of complex numbers and fix a pair of \( 2 \times 2 \) matrices \( A_s \) and \( B_t \) defined by

\[
(3.2) \quad A_s = \begin{pmatrix} z_1 & 0 \\ s\mu(z_1 - z_2) & z_2 \end{pmatrix}, \quad z_1, z_2 \in \Omega \text{ and } B_t = \begin{pmatrix} z & 0 \\ t\lambda & \bar{z} \end{pmatrix}, \quad z \in \Omega,
\]

where \( s, t \) are a pair of positive real numbers. Let

\[
(3.3a) \quad s_K = \frac{|K(z_1, z_2)|}{(K(z_1, z_1) K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}},
\]

\[
(3.3b) \quad t_K = \frac{K(z, z)}{(K(z, z)) [\|\partial_2 K(z, z)\|^2 - |\partial_2 K(z, z)\| |(\partial_2 K(z, z), K(z, z))|^2]^{1/2}}.
\]

It follows from the lemma that the matrix \( A_s \) (respectively, \( B_t \)) is the compression of the operator \( M \oplus M \) on the Hilbert space \( \mathcal{H}_K \oplus \mathcal{H}_K \) to the two-dimensional subspaces \( \mathcal{M} \) (respectively, \( \mathcal{N} \)) if and only if \( s = s_K \) and \( t = t_K \) and \( |\lambda| \leq 1 \).

A natural family of Hilbert spaces \( H^2_\alpha(\Omega) \) consisting of modulus automorphic holomorphic functions on \( \Omega \) was studied in the paper [2]. This family is indexed by \( \alpha \in \mathbb{T}^m \), where \( m \) is the number of bounded connected components in \( \mathbb{C} \setminus \Omega \) and \( \mathbb{T} \) is the unit circle. Each \( H^2_\alpha(\Omega) \) has a reproducing kernel \( K_\alpha(z, w) \). It was shown in [2] that every pure subnormal operator with spectrum \( \Omega \) and the spectrum of the normal extension contained in \( \partial \Omega \) is unitarily equivalent to \( M \) on one of these Hilbert spaces.

In the following subsection, we will show that any contractive homomorphism of the algebra \( \text{Rat}(\Omega) \) is of the form \( \rho_{A_s} \) or \( \rho_{B_t} \) with \( K = K_\alpha \) and \( |\mu| \leq 1 \) and \( |\lambda| \leq 1 \), respectively. Since the operator \( M \oplus M \) is subnormal, we will have exhibited the dilatation.

### 3.2. Construction of dilations

The generalization of the Nevanlinna-Pick theorem due to Abrahamse states that given \( n \) points \( w_1, w_2, \ldots, w_n \) in the open unit disk, there is a holomorphic function \( f : \Omega \to \mathbb{C} \) with \( f(z_i) = w_i \),
for $i = 1, 2, \ldots, n$ if and only if the matrix
\begin{equation}
(3.4) \quad M(w, \alpha) \overset{\text{def}}{=} (((1 - w_i \overline{w_j})K_{\alpha}(z_i, z_j)))
\end{equation}
is positive semidefinite. A deep result due to Widom (cf. [11, page 140]) shows that the map $\alpha \mapsto K_{\alpha}(z, w)$ is continuous for any fixed pair $(z, w)$ in $\Omega \times \Omega$.

In what follows, we shall first show that a homomorphism $\rho : \text{Rat}(\Omega) \to M_2$ is contractive if and only if it is of the form $\rho_{A_{\alpha}}$ or $\rho_{B_{\alpha}}$ with $|\mu| \leq 1$ and $|\lambda| \leq 1$, respectively and
\begin{equation}
(3.5a) \quad s = s_{\Omega}(z_1, z_2) := \sup \{|r(z_1)|^2 : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(z_2) = 0\}
\end{equation}
for any fixed but arbitrary pair $z_1, z_2 \in \Omega$, and
\begin{equation}
(3.5b) \quad t = t_{\Omega}(z) := \sup \{|r'(z)| : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(z) = 0\}
\end{equation}
for $z \in \Omega$.

We wish to point out that the extremal quantities $s_{\Omega}(z_1, z_2)$ and $t_{\Omega}(z)$ would remain the same even if we were to replace the space $\text{Rat}(\Omega)$ by the holomorphic function on $\Omega$. The solution to the first extremal problem, with holomorphic functions in place of $\text{Rat}(\Omega)$, exists by a normal family argument. Let $F : \Omega \to \mathbb{D}$ be a holomorphic function with $F(z_2) = 0$ and $F(z_2) = a$, where we have set $a = s_{\Omega}(z_1, z_2)$, temporarily. It then follows that $M((0, a), \alpha)$ must be non-negative definite for all $\alpha \in \mathbb{T}^m$. Consequently, we have
\begin{equation}
\det \begin{pmatrix}
K_{\alpha}(z_1, z_1) & K_{\alpha}(z_1, z_2) \\
K_{\alpha}(z_2, z_1) & (1 - a^2)K_{\alpha}(z_2, z_2)
\end{pmatrix} \geq 0
\end{equation}
for all $\alpha \in \mathbb{T}^m$. This condition is equivalent to requiring
\begin{equation}
|a|^2 \leq \frac{|K_{\alpha}(z_1, z_2)|^2}{K_{\alpha}(z_1, z_1)K_{\alpha}(z_2, z_2)} \leq 1 - \sup \left\{ \frac{|K_{\alpha}(z_1, z_2)|^2}{K_{\alpha}(z_1, z_1)K_{\alpha}(z_2, z_2)} : \alpha \in \mathbb{T}^m \right\}.
\end{equation}
As we have pointed out earlier, since $\alpha \to K_{\alpha}(z_i, z_j)$ is continuous for any pair of fixed indices $i$ and $j$, there exists a single $\alpha_0$ depending only on $z_1, z_2$ for which the supremum in the above inequality is attained. For this choice of $\alpha_0$ and $a$, clearly the determinant of $M((0, a), \alpha_0)$ is zero. It follows from [11, Theorem 4.4, p. 135] that the solution is unique and hence is a Blaschke product [11, Theorem 4.1, p. 130].

Similarly, the solution to the second extremal problem, with holomorphic functions in place of $\text{Rat}(\Omega)$, is a function which is holomorphic in a neighborhood of $\Omega$ [11, Theorem 1.6, p. 114]. Hence it is the limit of functions from $\text{Rat}(\Omega)$. The following Lemma first appeared as [13, Remark 2, p. 308].

**Lemma 3.** The homomorphism $\rho_{A_{\alpha}}$ is contractive if and only if $\|r(A_{\alpha})\| \leq 1$ for all $r$ in $\text{Rat}(\Omega)$ with $\|r\| \leq 1$ and $r(z_1) = 0$.
The homomorphism $\rho_{B_t}$ is contractive if and only if $\|r(B_t)\| \leq 1$ for all $r$ in $\text{Rat}(\Omega)$ with $\|r\| \leq 1$ and $r(z) = 0$.

Proof. The two proofs are similar, so we shall prove only (1). Suppose $r(A)$ is a contraction for all $r \in \text{Rat}(\Omega)$ with $\|r\| \leq 1$ and $r(z_1) = 0$. We have to prove that $r(A)$ is a contraction for all $r \in \text{Rat}(\Omega)$ with $\|r\| \leq 1$. For any such rational function $r$, let $r(z) = u$. Put $\varphi_u(z) = \frac{z + u}{1 - uz}$ and $\psi(z) = \varphi_u(r(z))$. Then $\psi$ is in $\text{Rat}(\Omega)$, $\|\psi\| \leq 1$ and $\psi(z) = 0$. By hypothesis, $\psi(A)$ is a contraction. Now note that $\varphi_u^{-1}(z) = \frac{z + u}{1 - uz}$. Since $\varphi_u^{-1}$ maps $\mathbb{D}$ into $\mathbb{D}$, by von Neumann’s inequality, $\|r(A)\| = \|\varphi_u^{-1}\psi(A)\| \leq 1$. \hfill $\Box$

This lemma makes it somewhat simple to derive the contractivity conditions for the homomorphisms induced by $A_s$ and $B_t$.

**Lemma 4.** The homomorphism $\rho_{A_s}$ is contractive if and only if

$$s^2 = s_\Omega(z_1, z_2)^{-1} - 1 \text{ and } |\mu| \leq 1.$$  

Similarly, the homomorphism $\rho_{B_t}$ is contractive if and only if

$$t = t_\Omega(z)^{-1} \text{ and } |\lambda| \leq 1.$$  

Proof. First, using the functional calculus for $A_s$, we see that

$$r \begin{pmatrix} z_1 \\ s\mu(z_1 - z_2) \\ z_2 \end{pmatrix} = \begin{pmatrix} r(z_1) \\ s\mu(r(z_1) - r(z_2)) \\ r(z_2) \end{pmatrix} = \begin{pmatrix} r(z_1) & 0 \\ s\mu r(z_1) & 0 \end{pmatrix},$$

assuming $r(z_2) = 0$. Therefore, contractivity of $\rho_{A_s}$ would imply

$$s^2 |\mu|^2 + 1 \leq (\sup \{ |r(z_2)|^2 : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(z_2) = 0 \})^{-1} = s_\Omega(z_1, z_2)^{-1}.$$  

Or, equivalently, if we put $s = s_\Omega(z_1, z_2)^{-1} - 1$, then we must have $|\mu| \leq 1$. Now an application of Lemma 3 completes the proof.

To obtain the contractivity condition for $\rho_{B_t}$, using the functional calculus, we see that

$$r \begin{pmatrix} z \\ t\lambda \\ z \end{pmatrix} = \begin{pmatrix} r(z) \\ t\lambda r'(z) \\ r(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a\lambda r'(z) \end{pmatrix},$$

assuming $r(z) = 0$. Therefore, contractivity of $\rho_{B_t}$ would imply that

$$t|\lambda| \leq (\sup \{ |r'(w)| : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(w) = 0 \})^{-1} = t_\Omega(z)^{-1}.$$  

Or equivalently, if we put $t = t_\Omega(z)^{-1}$, then we must have $|\lambda| \leq 1$. \hfill $\Box$

We now have enough material to construct the dilation for a homomorphism $\rho_T : \mathcal{A}(\Omega) \to \mathcal{M}_2$. In this case, $T$ is a $2 \times 2$ matrix with spectrum in $\Omega$. Since we can apply a unitary conjugation to make $T$ upper-triangular, it is enough to exhibit the dilation for the two matrices $T = A_s$ and $T = B_t$. 

3.3. Dilation for \( A_s \). Recall that there exists an \( \alpha_0 \) depending only on \( z_1 \) and \( z_2 \) such that \( \det M((0, a), \alpha_0) = 0 \). For now, set \( \alpha_0 = \alpha \). Let the subspace \( \mathcal{M} of H^2_\alpha \oplus H^2_\alpha \) be as in the first part of Lemma 2. For brevity, let

\[
m^2 = 1 - \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2)} > 0.
\]

Then

\[
\det M((0, m), \alpha) = \begin{pmatrix} K_\alpha(z_1, z_1) & K_\alpha(z_1, z_2) \\ K_\alpha(z_2, z_1) & (1 - m^2)K_\alpha(z_2, z_2) \end{pmatrix} = 0
\]

by the definition of \( m \). As we have pointed out earlier, there is a holomorphic function \( f : \Omega \to \mathbb{D} \) such that \( f(z_1) = 0 \) and \( f(z_2) = m \). Moreover, if \( g \) is any holomorphic function from \( \Omega \) to \( \mathbb{D} \) such that \( g(z_1) = 0 \), then the matrix \( M((0, g(z_2)), \alpha) \) is positive semidefinite, which implies that

\[
|g(z_2)|^2 \leq 1 - \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2)}.
\]

Thus

\[
m = \sup \left\{ |g(z_2)| : g \text{ is a holomorphic function from } \Omega \text{ to } \mathbb{D} \text{ and } g(z_1) = 0 \right\}.
\]

Hence

\[
s_\Omega(z_1, z_2)^{-1} - 1 = \frac{1}{m^2} - 1 = \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2) - |K_\alpha(z_1, z_2)|^2} = s_\mathbb{D}^2.
\]

So by the first part of Lemma 2, we have that the restriction of the operator \( M^* \oplus M^* \to \mathcal{M} \) in the orthonormal basis \( \{h_1(z_1, z_2), h_2(z_1, z_2)\} \) has the matrix representation \( A_s \) with \( s^2 = s_\mathbb{D}^2(z_1, z_2)^{-1} - 1 \) whenever \( |\mu| \leq 1 \).

3.4. Having constructed the dilation, it is natural to calculate the characteristic function, in the sense of Sz.-Nagy and Foias, when \( \Omega = \mathbb{D} \). In this case, the general form of the matrix \( T \) discussed above is

\[
T := \begin{pmatrix} z_1 & 0 \\ \mu(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2} & z_2 \end{pmatrix},
\]

where \( z_1 \) and \( z_2 \) are two points in the open unit disk \( \mathbb{D} \) and \( \mu \in \mathbb{C}, |\mu| \leq 1 \).

We are using the explicit value of \( s_\mathbb{D}(z_1, z_2) \) for the unit disc.

**Lemma 5.** For \( i = 1, 2 \), let \( \varphi_i(z) = (z - z_i)/(1 - \overline{z_i}z) \). The characteristic function of \( T \) is

\[
\theta_T(z) = \begin{pmatrix} (1 - |\mu|^2)^{1/2}\varphi_2(z) & -\mu \\ \overline{\mu}\varphi_1(z)\varphi_2(z) & (1 - |\mu|^2)^{1/2}\varphi_1(z) \end{pmatrix}.
\]
Proof. Recall that $\mathcal{M}$ is the subspace spanned by the orthonormal vectors $h_1(z_1, z_2)$ and $h_2(z_1, z_2)$. Since the compression of $M \oplus M$ to the co-invariant subspace $\mathcal{M}$ is $T$, by Beurling-Lax-Halmos theorem, we need to only find up to unitary coincidence (see [25], page 192, for definitions) the inner function whose range is $\mathcal{M}^\perp$. So let $(f \ g)$ be a vector in the orthogonal complement of $M$. The condition of orthogonality to $h_1$ implies that $g(z_1) = 0$, which is equivalent to $g = \varphi_1 \xi$ for arbitrary $\xi \in H^2(D)$. Now the orthogonality condition to $h_2$ implies that $(1 - |\mu|^2)\varphi_1(z_2)f(z_2) + \mu g(z_2) = 0$, which is the same as

$$ (3.8) \quad (1 - |\mu|^2)^{1/2} \varphi_1(z_2)f(z_2) + \mu g(z_2) = 0. $$

This implies that there is an $\eta_1 \in H^2(D)$ such that

$$ (1 - |\mu|^2)^{1/2} f + \mu g' = \varphi_2 \eta_1. $$

It is obvious that, conversely, if $(f \ g)$ is a function from $H^2(D) \oplus H^2(D)$ such that $g$ is in range of $\varphi$ and satisfies (3.8), then it is in the orthogonal complement of $M$.

Now let $\eta_2 = (1 - |\mu|^2)^{1/2} \xi - \bar{\mu} f$. Then

$$ \theta \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) = \left( \begin{array}{c} (1 - |\mu|^2)^{1/2} \varphi_2 \eta_1 - \mu \eta_2 \\ \bar{\mu} \varphi_1 \varphi_2 \eta_1 + (1 - |\mu|^2)^{1/2} \varphi_1 \eta_2 \end{array} \right) = \left( \begin{array}{c} f \\ g \end{array} \right). $$

Thus if $(f \ g)$ satisfies (3.8), then it is in the range of $\theta$. Conversely, it is easy to see that any element in the range of $\theta$ will satisfy (3.8). Thus the orthogonal complement of $M$ in $\mathcal{H}$ is the range of $\theta$. So $\theta$ is the characteristic function of the given matrix. \[\square\]

We would like to remark here that for $z_1 = z_2$ the characteristic function $\theta_T(u)$ for $T := \left( \begin{array}{cc} \lambda & 0 \\ \lambda(1 - |z|^2) & z \end{array} \right)$, $|\lambda| \leq 1$, can be obtained directly from the definition in case $z = 0$. A little computation, using the transformation rule for the characteristic function under a biholomorphic automorphism of the unit disk [25, pp. 239–240], produces the formula

$$ \theta_T(u) = \left( \begin{array}{cc} (1 - |\lambda|^2)^{1/2} \varphi(u) \\ \lambda \varphi^2(u) \end{array} \right), \quad u \in \mathbb{D} $$

in the general case.
Let $T_{\mu}$ be the matrix defined in (3.7). Note that if $T_{\mu'}$ and $T_{\mu}$ are two such matrices with $|\mu'| = |\mu|$, then
\[
\theta_{T_{\mu'}}(z) = \begin{pmatrix}
(1 - |\mu'|^2)^{1/2} \varphi_2 & -\mu' \\
\mu' \varphi_1 \varphi_2 & (1 - |\mu'|^2)^{1/2} \varphi_1
\end{pmatrix}
\]
\[= \begin{pmatrix}
(1 - |\mu|^2)^{1/2} \varphi_2 & -e^{i\psi} \mu \\
e^{-i\psi} \mu \varphi_1 \varphi_2 & (1 - |\mu|^2)^{1/2} \varphi_1
\end{pmatrix}
\text{for some } \psi \in [0, 2\pi]
\]
\[= \begin{pmatrix}
e^{i\psi/2} & 0 \\
0 & e^{i\psi/2}
\end{pmatrix}
\begin{pmatrix}
(1 - |\mu|^2)^{1/2} \varphi_2 & -\mu \\
\mu \varphi_1 \varphi_2 & (1 - |\mu|^2)^{1/2} \varphi_1
\end{pmatrix}
\begin{pmatrix}
e^{i\psi/2} & 0 \\
0 & e^{i\psi/2}
\end{pmatrix}
\]
\[= \begin{pmatrix}
e^{i\psi/2} & 0 \\
0 & e^{i\psi/2}
\end{pmatrix} \theta_{T_{\mu'}}(z) \begin{pmatrix}
e^{i\psi/2} & 0 \\
0 & e^{i\psi/2}
\end{pmatrix},
\]
and hence their characteristic functions coincide. So they are unitarily equivalent. Conversely, if $T_{\mu'}$ and $T_{\mu}$ are unitarily equivalent, then their characteristic functions coincide and hence the singular values of the characteristic functions are same. Note that when $z_1 \neq z_2$, we have
\[
\theta_{T_{\mu'}}(z_1)\theta_{T_{\mu'}}(z_1)^* = \begin{pmatrix}
(1 - |\mu'|^2)|\omega|^2 + |\mu'|^2 & 0 \\
0 & 0
\end{pmatrix}
\]
for some $\omega \in \mathbb{C}$ (independent of $\mu'$). When $z_1 = z_2$, then
\[
\theta_{T_{\mu'}}(z_1)\theta_{T_{\mu'}}(z_1)^* = \begin{pmatrix}
0 & |\mu'|^2 \\
0 & 0
\end{pmatrix}.
\]
In either case, coincidence of $\theta_{T_{\mu'}}$ and $\theta_{T_{\mu}}$ means that $|\mu'| = |\mu|$. Thus using the explicit characteristic function we have proved the following.

**Theorem 6.** Two matrices $T_{\mu'}$ and $T_{\mu}$ as defined in (3.7) are unitarily equivalent if and only if $|\mu'| = |\mu|$.

**3.5. Dilation for $B_t$.** We now shift our attention to the construction of dilation when the homomorphism $\rho_T$ is induced by a $2 \times 2$ matrix $T$ with equal eigenvalues. So $\sigma(T) = \{\zeta\}$. The domain $\Omega$ has its associated Szego kernel, which is denoted by $K_{\Omega}(z, w)$. Recall that a generalization due to Ahlfors to multiply connected domains of the Schwarz lemma says that
\[
\nu_{\Omega}(z) := (\sup \{|r'(z)| : r \in \text{Rat}(\Omega), |r| \leq 1 \text{ and } r(z) = 0\})^{-1} = \tilde{K}_{\Omega}(z, z)^{-1}.
\]
Let $\partial \Omega$ be the topological boundary of $\Omega$ and let $|dv|$ be the arc-length measure on $\partial \Omega$. Consider the measure $dm = |\tilde{K}_{\Omega}(\mu, z)|^2 |dv|$, and let the associated Hardy space $H^2(\Omega, dm)$ be denoted by $\mathcal{H}$. The measure $dm$ is mutually absolutely continuous with respect to the arc-length measure. Thus the evaluation functionals on $\mathcal{H}$ are bounded and hence $\mathcal{H}$ possesses a reproducing kernel $K$.

Then it is known that $K$ satisfies the property
\[
\tilde{K}_{\Omega}(z, z)^{-1} = \frac{K(z, z)}{(K(z, z)\|\partial_z K(\cdot, z)\|)^2 - (\|\partial_z K(\cdot, z), K(\cdot, z)\|^2)^{1/2}} = t_K;
\]
see [13, Theorem 2.2]. Now a (subnormal) dilation for \( B_t := \left( \lambda_t \frac{z_0}{z} \right) \), where \(|\lambda| \leq 1\), is the operator \( M \oplus M \) on the Hilbert space \( \mathcal{H} \oplus \mathcal{H} \). This is easily verified since the restriction of \( M^* \oplus M^* \) to the subspace \( \mathcal{N} \), which was described in the second part of Lemma 2, is \( B_t^* \).

**Remark 7.** If we choose \(|\mu| = 1\) then \( A_s^* \) is the restriction of \( M^* \) to the two-dimensional subspace spanned by the vectors \( K_\alpha(z_1, z) \) and \( K_\alpha(z_2, z) \) in the Hardy space \( H^2_\alpha(\Omega) \) by our construction. Except in this case, the dilation of the homomorphism \( \rho_{A_s} \) we have constructed is a minimal subnormal dilation. (This dilation then may be extended to a minimal normal dilation.) While it is known that a minimal dilation is not unique when \( \Omega \) is finitely connected, our construction gives a measure of this non-uniqueness. More explicitly, for each \( \alpha_0 \in \mathbb{T}^m \) for which

\[
\sup \left\{ \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2)} : \alpha \in \mathbb{T}^m \right\} = \frac{|K_{\alpha_0}(z_1, z_2)|^2}{K_{\alpha_0}(z_1, z_1)K_{\alpha_0}(z_2, z_2)},
\]

the matrix representation of the operator \( M^* \oplus M^* \) restricted to the 2-dimensional subspace \( \mathcal{M} \) of the Hilbert space \( H^2_{\alpha_0} \oplus H^2_{\alpha_0} \) equals \( A_s \).

4. The operator space

The problem that we are considering naturally gives rise to an operator space structure. In this section, we show that. We begin by recalling basic definitions.

A vector space \( X \) is called an operator space if for each \( k \in \mathbb{N} \) there are norms \( \| \cdot \|_k \) on \( X \otimes M_k \) satisfying:

1. Whenever \( A = ((a_{ij})) \in \mathcal{M}_k \), \( ((x_{ij})) \in X \otimes M_k \) and \( B = ((b_{ij})) \in \mathcal{M}_k \), then

\[
\| A \cdot ((x_{ij})) \cdot B \|_k \leq \| A \| \| ((x_{ij})) \|_k \| B \|
\]

where

\[
A \cdot ((x_{ij})) \cdot B = \left( \sum_{i=1}^{m} \sum_{j=1}^{k} a_{ij} x_{ip} b_{lj} \right) \in X \otimes \mathcal{M}_k
\]

and \( \|A\| \) and \( \|B\| \) are operator norms on \( \mathcal{M}_k = B(C^k) \).

2. For all positive integers \( m, k \) and for all \( R \in X \otimes \mathcal{M}_k \) and \( S \in X \otimes \mathcal{M}_m \), we have

\[
\left\| \left( \begin{array}{c} R \\ S \end{array} \right) \right\|_{m+k} = \max\{\| R \|_m, \| S \|_k \}.
\]

Two such operator spaces \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) are said to be completely isometric if there is a linear bijection \( \tau : X \to Y \) such that \( \tau \otimes I_k : (X, \| \cdot \|_X) \to (Y, \| \cdot \|_Y) \) is an isometry for every \( k \in \mathbb{N} \).

Let \( X \) be an operator space and let \( \rho : X \to B(\mathcal{H}) \) be a linear map, where \( \mathcal{H} \) is a Hilbert space. If for each \( k \in \mathbb{N} \) the map \( \rho \otimes I_k : (X, \| \cdot \|_k) \to B(\mathcal{H} \otimes \mathcal{M}_k) \)
is contractive then \(\rho\) is said to be completely contractive. Let \(\mathcal{H}\) be finite-dimensional, let \(T \in B(\mathcal{H})\), let \(X = \mathcal{A}(\Omega)\) and let \(\rho = \rho_T\) be as defined earlier. We assume that the eigenvalues \(z_1, z_2, \ldots, z_n\) of \(T\) are distinct.

To begin with, we introduce a notation. We denote by \(I^k_z\) the subset of \(\mathbb{C}^n \otimes \mathcal{M}_k\) defined as
\[
I^k_z = \{(R(z_1), R(z_2), \ldots, R(z_n)) : R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k \text{ and } \|R\| \leq 1\},
\]
where \(\|R\| = \sup_{z \in \Omega} \|R(z)\|\). When \(k = 1\), we denote it by \(I^1_z\).

**Lemma 8.** The set \(I^k_z\) defined above is a compact set.

**Proof.** Clearly, \(I^k_z\) is a subset of \(\overline{D}^n\). So it is enough to show that \(I^k_z\) is a closed set. Recall from Section 3 that the generalization of the Nevanlinna-Pick theorem due to Abrahamse states that given \(n\) points \(w_1, w_2, \ldots, w_n\) in the open unit disk, there is a holomorphic function \(f : \Omega \to \mathbb{C}\) with \(f(z_i) = w_i\) for \(i = 1, 2, \ldots, n\) if and only if the matrix
\[
M(w, \alpha) \overset{\text{def}}{=} ((1 - w_i \bar{w}_j)K_\alpha(z_i, z_j))
\]
is positive semidefinite for all \(\alpha \in \mathbb{T}^n\). So
\[
I^k_z = \{(w_1, w_2, \ldots, w_n) \in \mathbb{D}^n : \text{the matrix } M(w, \alpha) \text{ is positive semidefinite for all } \alpha \in \mathbb{T}^n\}
\]
\[
= \{(w_1, w_2, \ldots, w_n) \in \mathbb{D}^n : \lambda_{\min}(M(w, \alpha)) \geq 0 \text{ for all } \alpha \in \mathbb{T}^n\}
\]
\[
= \bigcap_{\alpha \in \mathbb{T}^n} \{(w_1, w_2, \ldots, w_n) \in \mathbb{D}^n : \lambda_{\min}(M(w, \alpha)) \geq 0\}
\]
\[
= \bigcap_{\alpha \in \mathbb{T}^n} (\lambda_{\min}(M(w, \alpha)))^{-1}([0, \infty)),
\]
where \(\lambda_{\min}(A)\) for a hermitian matrix \(A\) denotes its smallest eigenvalue. It is a continuous function on the set of hermitian matrices (see, for example, [7, Corollary III.2.6]). Thus \(w \mapsto \lambda_{\min}(M(w, \alpha))\) is a continuous function on \(\mathbb{C}^n\). Since an arbitrary intersection of closed sets is closed, \(I^k_z\) is a closed set. \(\square\)

It is easy to see that the set \(I^k_z\) is convex and balanced, so it is the closed unit ball of some norm on \(\mathbb{C}^n \otimes \mathcal{M}_k\). The sets of the form \(I^k_z\) were first studied, in the case \(k = 1\), by Cole and Werner [8]. The sets \(I^k_z\) are examples of matricially hyperconvex sets studied by Paulsen in [21]. Paulsen points out that the sequence of sets \(I^k_z \subseteq \mathbb{C}^n \otimes \mathcal{M}_k\) determines an operator space structure on \(\mathbb{C}^m\), that is, the set \(I^k_z\) determines a norm \(\|\cdot\|_{z,k}\) in \(\mathbb{C}^n \otimes \mathcal{M}_k\) such that \(I^k_z\) is the closed unit ball in this norm and the sequence \(\{\mathbb{C}^n \otimes \mathcal{M}_k, \|\cdot\|_{z,k}\}\) satisfies the conditions (1) and (2) above. We denote this operator space by \(\text{HC}_{\Omega,z}(\mathbb{C}^n)\). Paulsen also notes that this operator space is completely isometric to a quotient of a function algebra. Indeed, it is not difficult to see
that $HC_{\Omega,z}(\mathbb{C}^n)$ is completely isometrically isomorphic to the quotient of the operator algebra $A(\Omega)$ by $Z$, where

$$Z = \{ f \in A(\Omega) : f(z_1) = f(z_2) = \cdots = f(z_n) = 0 \}.$$ 

If $k = 1$, we will write $\| \cdot \|_{Z}$ rather than $\| \cdot \|_{Z,1}$.

**Lemma 9.** There are $n$ matrices $V_1, V_2, \ldots, V_n \in M_n$ such that the map $\rho_T \otimes I_k : A(\Omega) \otimes M_k \to M_n \otimes M_k$ is of the form

$$(\rho_T \otimes I_k)R = V_1 \otimes R(z_1) + V_2 \otimes R(z_2) + \cdots + V_n \otimes R(z_n)$$

for any $R \in A(\Omega) \otimes M_k$ and any $k \in N$. The matrices $V_i$ depend on the set \{z_1, z_2, \ldots, z_n\}.

**Proof.** If $F$ and $G$ are two elements of $A(\Omega) \otimes M_k$ which agree on the set \{z_1, z_2, \ldots, z_n\}, then define $H \in A(\Omega) \otimes M_k$ by $H = F - G$. Then $H$ vanishes at the points $z_1, z_2, \ldots, z_n$ and hence $H(z) = (z - z_1)(z - z_2) \cdots (z - z_n)W(z)$ for some $W$ in $A(\Omega) \otimes M_k$. By the functional calculus,

$$(\rho_T \otimes I_k)H = (T - z_1)(T - z_2) \cdots (T - z_n)W(T).$$

Note that $(z - z_1)(z - z_2) \cdots (z - z_n)$ is the characteristic polynomial of $T$ and by Cayley-Hamilton theorem, $(T - z_1)(T - z_2) \cdots (T - z_n) = 0$. Thus $(\rho_T \otimes I_k)H = 0$. So if $F, G \in A(\Omega) \otimes M_k$ are such that $F(z_i) = G(z_i)$ for all $i = 1, 2, \ldots, n$, then $(\rho_T \otimes I_k)F = (\rho_T \otimes I_k)G$. Now define $V_1, V_2, \ldots, V_n$ by

$$V_i = \rho_T \left( \frac{(z - z_1) \cdots (z - z_i - 1)(z - z_{i+1}) \cdots (z - z_n)}{(z_i - z_1) \cdots (z_i - z_i - 1)(z_i - z_{i+1}) \cdots (z_i - z_n)} \right)$$

for $i = 1, 2, \ldots, n$. Given $R \in A(\Omega) \otimes M_k$, it agrees with the function

$$\tilde{R}(z) = \sum_{i=1}^n \frac{(z - z_1) \cdots (z - z_i - 1)(z - z_{i+1}) \cdots (z - z_n)}{(z_i - z_1) \cdots (z_i - z_i - 1)(z_i - z_{i+1}) \cdots (z_i - z_n)} R(z_i)$$

on the set \{z_1, z_2, \ldots, z_n\} and hence

$$(\rho_T \otimes I_k)R = (\rho_T \otimes I_k)\tilde{R}$$

$$= \sum_{i=1}^n \rho_T \left( \frac{(z - z_1) \cdots (z - z_i - 1)(z - z_{i+1}) \cdots (z - z_n)}{(z_i - z_1) \cdots (z_i - z_i - 1)(z_i - z_{i+1}) \cdots (z_i - z_n)} \right) \otimes R(z_i)$$

$$= \sum_{i=1}^n V_i \otimes R(z_i)$$

completing the proof of the lemma. \hfill \square

The referee points out that $V_i^2 = V_i$, $1 \leq i \leq n$, and $V_1 + \cdots + V_n = I_n$. In particular, $V_1, \ldots, V_n$ of the preceding lemma cannot be arbitrary.

At this point, we note that $A(\Omega)$ being a closed sub-algebra of the commutative $C^*$-algebra of all continuous functions on the boundary of $\Omega$ inherits a natural operator space structure, denoted by $\text{MIN}(A(\Omega))$. Recall
that a celebrated theorem of Arveson says that a contractive homomorphism \( \rho_T : A(\Omega) \to L(H) \) dilates if and only if it is completely contractive when \( A(\Omega) \) is equipped with the MIN operator space structure. The contractivity and complete contractivity of the homomorphism \( \rho_T \) amount to, respectively,

\[
\sup \{ \| w_1 V_1 + \cdots + w_n V_n \| : w = (w_1, \ldots, w_n) \in I_z \} \leq 1,
\]

where \( \| \cdot \| \) is the operator norm on \( M_n \), and

\[
\sup \left\{ \left\| \sum_{i=1}^n V_i \otimes W_i \right\| : W_i \in M_k \text{ and } W = (W_1, \ldots, W_n) \in I_k \right\} \leq 1,
\]

where \( \| \cdot \| \) is the operator norm on \( M_n \otimes M_k \). Now, we state the following theorem whose proof is evident from the discussion above.

**Theorem 10.** The contractive homomorphism \( \rho_T : A(\Omega) \to M_n \) is completely contractive with respect to the MIN operator space structure on \( A(\Omega) \) if and only if the contractive linear map \( L_T : (C^n, \| \cdot \|_z) \to M_n \) defined by \( L_T(w) = w_1 V_1 + w_2 V_2 + \cdots + w_n V_n \) is completely contractive on the operator space \( HC_{\Omega,z} (C^n) \).

The theorem above brings us to our concluding remarks of this section. Given a Banach space, there are two extremal operator space structures on it, denoted by MAX(\( X \)) and MIN(\( X \)). We refer the reader to [20] for definitions and basic details. However, this theorem shows that if \( HC_{\Omega,z} (C^n) \) is completely isometric to MAX(\( C^n, \| \cdot \|_z \)), then every contractive homomorphism \( \rho_T \) of the algebra \( A(\Omega) \), induced by an \( n \)-dimensional linear transformation \( T \) with distinct eigenvalues in \( \Omega \), will necessarily dilate. This gives rise to the question of determining when \( HC_{\Omega,z} (C^n) \) is the same as MAX(\( C^n, \| \cdot \|_z \)), which is an interesting question in its own right.

In [20], Paulsen related a problem similar to the one that we are considering to certain questions in the setting of operator spaces and thereby solved it. For \( n \geq 1 \), let \( G \) be a closed unit ball in \( C^n \) corresponding to a norm \( \| \cdot \|_G \) on \( C^n \). Let \( A(G) \) denote the closure of polynomials in \( C(G) \), the algebra of all continuous functions on \( G \) equipped with the sup norm. He showed that if MIN(\( C^n, \| \cdot \|_G \)) is not completely isometric to MAX(\( C^n, \| \cdot \|_z \)), then there exists a unital contractive homomorphism \( \rho : A(G) \to B(H) \), for some Hilbert space \( H \), which is not completely contractive. Paulsen proved the remarkable result that for \( n \geq 5 \),

\[
\text{MIN}(C^n, \| \cdot \|_G) \text{ is not completely isometric to } \text{MAX}(C^n, \| \cdot \|_G),
\]

for any closed unit ball \( G \). For \( n = 2 \) and \( G = D^2 \), Ando’s theorem implies that \( \text{MIN}(C^2, \| \cdot \|_{D^2}) \) is completely isometric to \( \text{MAX}(C^2, \| \cdot \|_{D^2}) \). The fact that (4.4) holds for \( n = 3 \) and any closed unit ball \( G \) is pointed out in [22, Exercise 3.7].
In the same spirit, a similar question about a class of homomorphisms, first introduced by Parrott [19] (see also [14], [15] and [16]), led Paulsen to define a natural operator space, which he called COT. Let $G$ be a unit ball and let $w$ be a point in the interior of $G$. Let $X$ be the Banach space $X = (C^n, \|\cdot\|_{G,w})$, where $\|\cdot\|_{G,w}$ is the Caratheodory norm of $G$ at the point $w$. The question of whether $\text{COT}_w(X)$ is completely isometric to $\text{MIN}(X^*)$ for $w \in G$ was first raised in [20]. Paulsen showed that the answer is affirmative when $w = 0$. Later in an unpublished note, it was shown by Dash [9] that $\text{COT}_w(G)$ and $\text{MIN}(X^*)$ are not necessarily completely isometric. The question of deciding whether a contractive homomorphism $\rho_T : A(\Omega) \rightarrow B(H)$ is completely contractive or not is similar in nature. It amounts to deciding if $HC_{\Omega,z}(C^n)$ is completely isometric to $\text{MIN}(C^n, \|\cdot\|_z)$ or $\text{MAX}(C^n, \|\cdot\|_z)$. It is likely that the operator space $HC_{\Omega,z}(C^n)$ is completely isometric to $\text{MIN}(C^n, \|\cdot\|_z)$ for every $n \geq 3$. We pose this as an open problem whose solution defies us at the moment.

5. A factorization condition

Let $T$ be a linear transformation on an $n$-dimensional Hilbert space $V$ with distinct eigenvalues $z_1, z_2, \ldots, z_n$ in $\Omega$. Let $v_1, v_2, \ldots, v_n$ be the $n$ linearly independent eigenvectors of $T^*$. If $\sigma = \{z_1, z_2, \ldots, z_n\}$, then define a positive definite function $K : \sigma \times \sigma \rightarrow \mathbb{C}$ by setting

$$K(z_j, z_i) := \left\langle v_i, v_j \right\rangle,$$

As before, let $\rho_T : A(\Omega) \rightarrow L(V)$ be the homomorphism induced by $T$. Suppose there exists a dilation of the homomorphism $\rho_T$. Then it follows from [1, Theorem 2] that there is a flat unitary vector bundle $E$ of rank $n$ (see [2] for definitions and complete results on the model theory in multiply connected domains) such that $\rho_T(f)$ is the compression of the subnormal operator $M_f$ on the Hardy space $H^2_E(\Omega)$ to a semi-invariant subspace in it. Consequently, for some choice of a flat unitary vector bundle $E$, the homomorphism

$$\rho_M : A(\Omega) \rightarrow B(H^2_E(\Omega))$$

dilates $\rho_T$. The homomorphism $\rho_M$ is induced by the multiplication operator $M$ on $H^2_E(\Omega)$, which is subnormal. Thus the homomorphism $\rho_N : C(\partial \Omega) \rightarrow B(\mathcal{H})$ induced by the normal extension $N$ on the Hilbert space $\mathcal{H} \supseteq H^2_E(\Omega)$ of the operator $M$ is a dilation of the homomorphism $\rho_T$ in the sense of (1.1). The multiplication operator $M$ on $H^2_E$ is called a bundle shift. We recall [2, Theorem 3] that $\dim \ker(M - z)^* = n$. Let $K^E(z_i)$, $i = 1, 2, \ldots, n$, be a basis (not necessarily orthogonal) of $\ker(M - z)^*$. We set

$$K^E(z_j, z_i) := \left\langle (K^E_z(\ell), K^E_z(p)) \right\rangle_{\ell,p=1}^n,$$

for $1 \leq i, j \leq n$. 


If $\rho_T$ dilates, then the linear transformation $T$ can be realized as the compression of the operator $M$ on $H^2_2(\Omega)$ to an $n$-dimensional co-invariant subspace, say $\mathcal{M} \subseteq H^2_2(\Omega)$. The subspace $\mathcal{M}$ must consist of eigenvectors of the bundle shift $M$. Let $x_i$, $1 \leq i \leq n$, be a set of $n$ vectors in $\mathbb{C}^n$ and

$$\mathcal{M} = \left\{ \sum_{\ell=1}^{n} x_i(\ell)K^E(\ell) : 1 \leq i \leq n \right\}.$$  

The map which sends $v_i$ to $\sum_{\ell=1}^{n} x_i(\ell)K^E(\ell)$, $1 \leq i \leq n$, intertwines $T^*$ and the restriction of $M^*$ to $\mathcal{M}$. For this map to be an isometry as well, we must have

$$\langle v_i, v_j \rangle = \langle K^E(z_j, z_i)x_i, x_j \rangle, \quad x_i \in \mathbb{C}^n, \quad 1 \leq i, j \leq n. \quad (5.4)$$

Conversely, if there is a flat unitary vector bundle $E$ and $n$ vectors $x_1, \ldots, x_n$ in $\mathbb{C}^n$ satisfying (5.4), then $\rho_T$ obviously dilates. So we have proved the following theorem.

**Theorem 11.** The homomorphism $\rho_T$ is dilatable in the sense of (1.1) if and only if the kernel $K$, as defined in (5.1), can be written as

$$K(z_j, z_i) = \langle K^E(z_j, z_i)x_i, x_j \rangle, \quad \text{for some choice of } x_1, \ldots, x_n \in \mathbb{C}^n,$$

and some choice of a flat unitary vector bundle $E$ of rank $n$.

It is interesting to see how contractivity of $\rho_T$ is related to the above theorem. Note that $\rho_T$ is contractive if and only if $\|f(T)^*\| \leq \|f\|$ by the definition of $\rho_T$. Since $T^*v_i = \bar{z}_i v_i$, we note that $f(T)^*v_i = \bar{f}(z_i)v_i$, for $1 \leq i \leq n$ and $f \in \text{Rat}(\Omega)$. It then follows that

$$\|\rho_T(f)^*\|^2 = \sup \left\{ \|f(T)^* \left( \sum_{i=1}^{n} \alpha_i v_i \right) \|^2 : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \right\}$$

$$= \sup \left\{ \| \sum_{i=1}^{n} \alpha_i f(z_i)v_i \|^2 : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \right\}$$

$$= \sup \left\{ \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j f(z_i)f(z_j)\langle v_i, v_j \rangle : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \right\}. \quad \text{(5.5)}$$

Therefore, $\|f(T)^*\| \leq \|f\|$ if and only if

$$\sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j f(z_i)f(z_j)\langle v_i, v_j \rangle \leq \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j \langle v_i, v_j \rangle,$$

for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and all $f \in \text{Rat}(\Omega)$ with $\|f\| \leq 1$. Thus contractivity of $\rho_T$ is equivalent to non-negative definiteness of the matrix

$$\left( (1 - \bar{f}(z_i)f(z_j))K(z_j, z_i) \right)_{i,j=0}^{n}.$$
for all \( f \in \text{Rat}(\Omega) \), \( \|f\| \leq 1 \). If \( \rho_T \) is dilatable then the theorem above tells us that
\[
(5.6) \quad (1 - \overline{f(z_i)}f(z_j))K(z_j, z_i) = (1 - \overline{f(z_i)}f(z_j))\langle K_\alpha(z_j, z_i)x_i, x_j \rangle_{i,j=0}^n.
\]
The last matrix is non-negative definite because \( M \) on \( \mathcal{H}_2(\Omega) \) induces a contractive homomorphism. We therefore see, in this concrete fashion, that if the homomorphism \( \rho_T \) were dilatable then it would be contractive.

The interesting point to note here is that our construction of the dilation when \( T \) is a \( 2 \times 2 \) matrix proves that the general dilation in that case is of the form \( \mathcal{H}_2^\alpha(\Omega) \otimes \mathbb{C}^2 \).

Suppose that the homomorphism \( \rho_T \) admits a dilation of the form
\[
(5.7) \quad \rho_{M \otimes I} : \mathcal{A}(\Omega) \to \mathcal{B}(\mathcal{H}_2^\alpha(\Omega) \otimes \mathbb{C}^n)
\]
for some \( \alpha \in \mathbb{T}^n \), that is, the multiplication operator \( M \otimes I \) on \( \mathcal{H}_2^\alpha(\Omega) \otimes \mathbb{C}^n \) is a dilation of \( T \). Since the eigenvectors \( \{v_1, v_2, \ldots, v_n\} \) for \( T^* \) span \( V \) and the set of eigenvectors of \( M^* \otimes I : \mathcal{H}_2^\alpha(\Omega) \otimes \mathbb{C}^n \to \mathcal{H}_2^\alpha(\Omega) \otimes \mathbb{C}^n \) at \( z_i \) is the set of vectors \( \{K_\alpha(\cdot, z_i) \otimes a_j : a_j \in \mathbb{C}^n, 1 \leq j \leq n\} \) for \( 1 \leq i \leq n \), it follows that any map \( \Gamma : V \to \mathcal{H}_2^\alpha(\Omega) \) that intertwines \( T^* \) and \( M^* \) must be defined by \( \Gamma(v_i) = K_\alpha(\cdot, z_i) \otimes a_i \) for some choice of a set of \( n \) vectors \( a_1, a_2, \ldots, a_n \) in \( \mathbb{C}^n \). Now \( \Gamma \) is isometric if and only if
\[
(5.8) \quad (K(z_j, z_i)) = ((v_i, v_j)) = (K_\alpha(z_j, z_i)\langle a_i, a_j \rangle).
\]
Clearly, this means that \( (K(z_j, z_i)) \) admits \( (K_\alpha(z_j, z_i)) \) as a factor in the sense that \( (K(z_j, z_i)) \) is the Schur product of \( (K_\alpha(z_j, z_i)) \) and a positive definite matrix, namely, the matrix \( A = ((a_i, a_j)) \).

Conversely, the contractivity assumption on \( \rho_T \) does not necessarily guarantee that \( K_\alpha \) is a factor of \( K \). However, if we make this stronger assumption, that is, we assume there exists a positive definite matrix \( A \) such that \( (K(z_j, z_i)) = (K_\alpha(z_j, z_i)a_{ij}) \), where \( A = ((a_{ij})) \). Since \( A \) is positive, it follows that \( A = ((a_i, a_j)) \) for some set of \( n \) vectors \( a_1, \ldots, a_n \) in \( \mathbb{C}^n \). Therefore if we define the map \( \Gamma : V \to \mathcal{H}_2^\alpha(\Omega) \otimes \mathbb{C}^n \) to be \( \Gamma(v_i) = K_\alpha(\cdot, z_i) \otimes a_i \) for \( 1 \leq i \leq n \) then \( \Gamma \) is clearly unitary and is an intertwiner between \( T \) and \( M^* \). Thus the theorem above has the following corollary:

**Corollary 12.** The homomorphism \( \rho_T \) is dilatable to a homomorphism \( \hat{\rho} \) of the form (5.7) if the kernel \( K_\alpha \), as defined in (5.1), is the Schur product of a positive definite matrix \( A \) and the restriction of \( K_\alpha \) to the set \( \sigma \times \sigma \) for some \( \alpha \in \mathbb{T}^n \).

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