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Theoretical modelling of laminated composite plates

A V KRISHNA MURTY

Department of Aerospace Engineering, Indian Institute of Science, Bangalore 560 012, India

Abstract. Formulation of appropriate governing equations, simpler than the three-dimensional equations of elasticity yet capable of predicting, fairly accurately, all important response parameters such as stress and strain, is attempted in modelling a structural component. Several theoretical models are available in the literature for the analyses of plates. The emergence of fibre-reinforced plastics as an attractive form of structural construction, added a new complexity to the modelling considerations of laminates by requiring the estimation of the interlaminar stresses and strains. In this paper, modelling considerations of laminated composite plates are discussed. The classical laminated plate theory and higher-order shear deformation models are reviewed to bring out their interlaminar stress predictive capabilities, and some new modelling possibilities are indicated.

Keywords. Laminated composite plates; analyses of plates; fibre-reinforced plastics; interlaminar stress and strain; shear deformation models.

1. Introduction

In general one may define a structure as a material system intended to carry loads, and the structural element as its component. Structural elements are usually categorized for convenience into beams, plates, shells, plane problems etc. depending upon the geometry and loading pattern. The set of equations, whose solution can be interpreted to get response parameters such as stresses and strains, describing the behaviour of the structural element is referred to here as the theoretical model or simply the model. If rigour is the only criterion, the three-dimensional elasticity model is the only valid formulation for all these categories of problems. The complexity of three-dimensional equations, defying the possibility of any reasonable solution for a large class of practical problems, motivated the search for simpler sets of equations, each applicable for a particular structural element and capable of yielding all important response parameters reasonably accurately. Such solutions will naturally not coincide with the three-dimensional solution in all details. The aim is to formulate these simpler sets of equations such that they yield results as close to the solution of the three-dimensional equations of elasticity as possible. Simplicity and accuracy, although contradictory, have been the favourite objectives in shaping theoretical models. Identification and incorporation of appropriate assumptions, reflecting the physical behaviour of a structural element, into the three-dimensional theory guided the development of theoretical models. The simple assumption of Kirchhoff, that normals to the midsurface of a plate before deformation will remain normal even after deformation, brought about far-reaching simplicity in the model and shaped the plate theory.

In what follows, we will be concerned mainly with modelling of plates. Within the frame work of the three-dimensional theory of elasticity, a plate subjected to transverse loading requires the solution of three simultaneous partial differential equations or its equivalent along with appropriate boundary conditions. On the other hand the classical plate model, based on Kirchhoff's assumption predicts plate behaviour fairly accurately as the solution of the biharmonic equation in normal deflection with appropriate boundary conditions. As a consequence of the simplification, some inconsistencies arise. Some of them are

(i) transverse shear and normal strains are zero,
(ii) constitutive law is violated,
(iii) there is no provision for adequate description of boundary conditions at edges,
(iv) there is no provision for considering boundary conditions on plate surfaces.

Recognizing the importance of the transverse shear strain in thick plates, shear deformation theories with provision for non-zero transverse shear strain were developed to extend the theory to thick plates (Reissner 1944; Mindlin 1951). The problem of providing adequate description of the boundary conditions at edges is resolved, once the shear strains are accomodated. Unfortunately there appears to be no general way to model plates satisfying plate surface boundary conditions. Krishna Murty (1977), Levinson (1980), Krishna Murty & Vellaichamy (1988a) attempt to model the plates satisfying the zero transverse shear strain conditions at plate surface and Vijaya Kumar & Krishna Murty (1988a) examine the modelling of plates satisfying the normal stress condition at the plate surface also, using the Lagrangian multipliers. These typical developments in modelling isotropic plates provide a basis for modelling laminated plates.

The advent of composites as the most attractive engineering material system, forced a fresh look at the modelling considerations of laminates. In these materials the interlaminar zone is the weakest link as it is essentially a thin layer of homogeneous resin medium. The estimation of interlaminar stresses and strains has became essential in ensuring laminate integrity. Thus a fresh look at the plate theory, becomes necessary to expose the interlaminar stress predictive capabilities of existing models and, if required, to remodel the laminates.

We shall restrict our further discussion to the modelling of fibre-reinforced plastic laminates. Modern fibre-reinforced plastic materials are essentially bundles of fibres embedded in resin. The fibre strength is very high compared to the resin. Diameters of the fibres are of the order of microns. When prepreg material is used to build structural components, which is perhaps the most popular method of building composite laminates currently, we have layers or plies of unidirectional fibres embedded in resin, in a partially polymerized form. A number of such plies are put together to obtain the necessary thickness and are cured under controlled temperature and pressure conditions to build the structural components. Therefore, it is necessary to fix up the type of abstraction for modelling the laminates. Two levels are readily apparent. In order to consider localized details such as the stress field around a fibre, fibre matrix

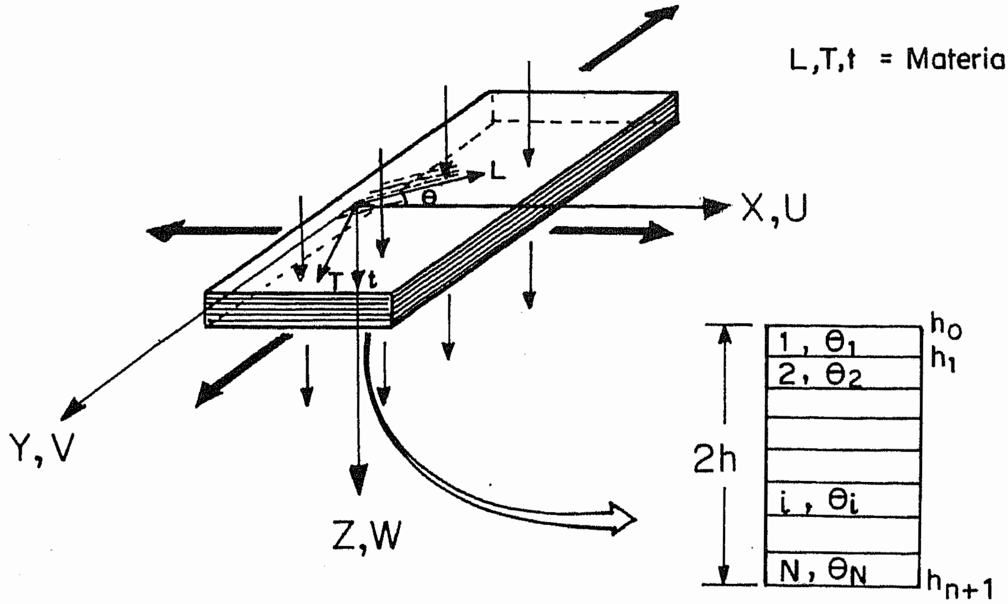

Figure 1. A typical laminate – its geometry and coordinates.

interactions etc., no simplifications are possible and full elasticity equations are the obvious choice. On the other hand, by restricting the attention to certain gross aspects, such as ply and interlaminar strength, one may consider the laminate as a number of layers of an orthotropic homogeneous medium, perfectly bonded at the surfaces such that no slip is possible. Here we consider the second abstraction, and generally, as is well-known, this kind of abstraction provides valuable information of direct use in engineering applications, in assessing the strength, stiffness and vibrational behaviour of laminates.

A typical laminate is shown in figure 1. A look at some of the exact solutions available (Pagano 1970), will reveal the special complexity in modelling laminates. Unlike metals, laminates contain interfaces across which the material constants are discontinuous in the direction of thickness. As a consequence, some of the strain and stress components are discontinuous across the interface, as indicated in table 1.

We shall refer to (ε_x, ε_y and ε_{xy}) and (σ_x, σ_y, σ_{xy}) as in-plane strains and stresses, respectively and (ε_z ε_{zx} ε_{zy}) and (σ_z σ_{zx} and σ_{zy}) as transverse strains and stresses, respectively. It may be noted here that in-plane strains are continuous across the interface while the corresponding stresses are discontinuous. Similarly transverse stresses are continuous across the interface while the corresponding strains are discontinuous across the interface. Generally it is difficult to achieve this feature in pure displacement- or stress-based models. We will see later that in the iterative model described later this feature has been realized.

Recognizing that the formulation of the governing equations of a structural element

Table 1. Nature of stress and strain components across the interfaces.

Continuous	ε_x	ε_y	ε_{xy}	σ_z	σ_{zx}	σ_{zy}
Discontinuous	σ_x	σ_y	σ_{xy}	ε_z	ε_{zx}	ε_{zy}

has a basis in the energy principles, it may be noted that the approach for modelling is essentially the Kantorovich form of the Rayleigh-Ritz method. Two classical approaches will be readily evident. In the first one the displacement is expressed in a series form in the thicknesswise or z-coordinate, leaving the rest in a functional form, and the governing equations are established retaining a few appropriate terms. We shall call such models here 'the displacement-based models'. The second approach treats stresses as primary variables. Soni & Pagano (1988) discuss an interesting new model based on this approach in this volume and the present discussion will be limited to displacement models.

2. Displacement-based models

The procedure consists of expanding the displacements in terms of the thicknesswise coordinate as,

$$U(x,y,z)=\sum_{i=0,1,2} z^i u_i(x,y),$$
$$V(x,y,z)=\sum_{i=0,1,2} z^i v_i(x,y),$$
$$W(x,y,z)=\sum_{i=0,1,2} z^i w_i(x,y). \tag{1}$$

Retaining a few terms and utilizing the energy principle or direct equilibrium considerations, the governing equations and boundary conditions are established (Reissner & Stavsky 1961; Dong *et al* 1962; Yang *et al* 1966; Whitney & Leissa 1969; Whitney & Pagano 1970; Whitney & Sun 1973; Nelson & Lorch 1974; Lo *et al* 1977). Models based on (1) have no provision for satisfaction of the plate surface boundary conditions. In general, surfaces of the laminate may receive both normal and tangential loads. A general method for modelling plates satisfying the plate surface conditions, as applied to isotropic plates is discussed by Vijaya Kumar & Krishna Murty (1988a). In this study Lagrangian multipliers were used to satisfy the normal stress condition at the plate surface. The results did not indicate significant improvement in the accuracies in predicting stresses, commensurate with the complexity of the formulation. In practice plates are usually subjected to normal surface loads only, and in that case, shear strains at plate surfaces are zero. This feature can be incorporated into the models much more easily by rewriting the expansion for displacements in the form,

$$U(x,y,z)=\sum_{i=0,1,2} p_i(\xi)u_i(x,y),$$
$$V(x,y,z)=\sum_{i=0,1,2} p_i(\xi)v_i(x,y),$$
$$W(x,y,z)=\sum_{i=0,1,2} q_i(\xi)w_i(x,y), \tag{2}$$

where

$$p_0=1,\quad q_0=1,\quad p_1=\xi,\quad q_1=\xi,\quad p_2=\xi^2/2,$$
$$p_n=\xi(1-\xi^{n-1}/n),\qquad n=3,5,7\ldots,$$
$$q_n=1-\xi^n,\qquad n=2,4,6\ldots,$$
$$p_n=\xi^2(1-2\xi^{n-2}/n),\quad n=4,6,8\ldots,$$

$$q_n = \xi(1 - \xi^{n-1}), \qquad n = 3, 5, 7\ldots, \tag{3a}$$

$$\left.\begin{aligned} u_1 &= -w_{0,x} \\ v_1 &= -w_{0,y} \end{aligned}\right\},$$

$$\left.\begin{aligned} u_2 &= -w_{1,x} \\ v_2 &= -w_{1,y} \end{aligned}\right\}, \tag{3b}$$

and

$$\xi = z/h. \tag{3c}$$

Note that u_1 and v_1 are related to w_0 as in (3b) to accomodate Kirchhoff's model as a special case. This represents a generalization of several models currently available in the literature (Murthy 1981; Bhimaraddi & Stevens 1984; Reddy 1984, 1987; Krishna Murty 1987, 1988; Krishna Murty & Harikumar 1988; Krishna Murty & Vellaichamy 1988b).

In the displacement-based models all stresses can be estimated using the constitutive relations. We shall call such estimates of stresses as "the direct estimates". However, since the material constants are discontinuous across the interfaces, direct estimates of all stresses will be discontinuous across the interface. Therefore it is difficult to realize the true nature of interlaminar stresses as they are not continuous across the interfaces (see table 1). Thus direct estimates of interlaminar stresses are unlikely to be accurate. On the other hand, since in-plane stresses $\sigma_x \sigma_y$ and σ_{xy} are discontinuous across interfaces, their direct estimates can be expected to be accurate. A $[0/90]_s$ laminated plate strip, infinitely long y-axis simply supported at opposite edges at $x = 0$, a, subjected to sinusoidal load

$$(\sigma_z)_{z=\pm h} = \pm q_0 \sin(\pi x/a)$$

was analysed in detail by Krishna Murty & Vellaichamy (1988b) using the higher order shear deformation theory (HOST) and the classical laminated plate theory (CLPT). All the four layers of the laminate have equal thickness and the total plate thickness is $2h$. The middle surface corresponds to $\xi = 0$, and $\xi = 0.5$ represents the interface. Material constants of the 0° layer are taken to be $E_L/E_T = 25$, $E_T/E_t = 1$, $G_{LT}/E_T = 0.5$, $G_{LT} = G_{Lt}$, $G_{Tt}/E_T = 0.2$, $\nu_{LT} = \nu_{Tt} = \nu_{tL} = 0.25$. A comparison of a typical in-plane stress normalized with the applied load $\sigma_x^* = \sigma_x/q_0$ is shown in table 2.

The exact solution is based on the theory of elasticity. It is clear that the σ_x^* is estimated fairly accurately in HOST and the discontinuity at the interface is revealed

Table 2. Comparison of σ_x^* $(2h/a = 0.08)$.

ξ	Exact	Direct estimates	
		HOST	CLPT
0·2	1·46	1·52	1·83
0·4	2·93	3·13	3·67
0·5	3.67	4·00	4·59
	87·85	98·45	107·90
0·6	115·40	121·21	129·48
0·8	173·50	172·03	172·65
1·0	236·95	231·79	215·80

Table 3. Comparison of σ_z^* ($2h/a = 0{\cdot}08$).

ξ	Exact	Direct estimate HOST	Statically equivalent estimate HOST	Statically equivalent estimate CLPT
0·2	0·255	0·080	0·257	0·259
0·4	0·509	0·184	0·513	0·519
0·5	0·636	{0·252, 0·504}	0·640	0·646
0·6	0·754	0·647	0·758	0·763
0·8	0·932	1·001	0·934	0·936
1·0	1·000	1·475	1·00	1·0

satisfactorily. Further we see that the estimates by CLPT, contain a error of about 10% only in the maximum value of σ_x^*, even when the thickness to width ratio is as large as 0·08. Comparison of a typical interlaminar stress, $\sigma_z^* = \sigma_z/q_0$, with the exact solution is shown in table 3.

The direct estimates indicate an unrealistic discontinuity at the interface and contain an error of 47.5% in the maximum value of the normal stress at the plate surface. Recognising that the governing equations in these models represent elemental equilibrium in an average sense, it may be noted that the equations of equilibrium, representing pointwise equilibrium,

$$\sigma_{ij,j} = 0, \quad i, j = x, y, z, \tag{4}$$

are available to obtain better estimates of interlaminar stresses than the direct estimates.

Substituting the in-plane estimates for stresses σ_x, σ_y and σ_{xy} in the equations of equilibrium (4) and integrating, one can get estimates to interlaminar stresses as,

$$\sigma_{xz} = -\int(\sigma_{x,x} + \sigma_{xy,y})\,\mathrm{d}z + \text{constant},$$

$$\sigma_{yz} = -\int(\sigma_{y,y} + \sigma_{xy,x})\,\mathrm{d}z + \text{constant},$$

$$\sigma_z = -\int(\sigma_{xz,x} + \sigma_{yz,y})\,\mathrm{d}z + \text{constant}. \tag{5}$$

Such estimates are referred to here as "statically equivalent estimates". We see from table 3 that such estimates for interlaminar stresses display the required continuity across the interface and agree closely with the exact solution. It may be noted that the statically equivalent estimates of interlaminar stresses by the classical plate theory are also close to the exact solution. Table 4 shows a comparison of the interlaminar strain,

$$\varepsilon_z^* = \varepsilon_z/\varepsilon_0,$$

where $\varepsilon_0 = (2/D_{11}\pi^2)(a/h)^2$ and D_{11} is the plate bending rigidity.

Direct estimates display an unrealistic continuity across the interface $\xi = 0{\cdot}5$. The statically equivalent estimates of strain are obtained using the statically equivalent estimates of interlaminar stresses and direct estimates of in-plane stresses in the constitutive relations. These statically equivalent estimates of strains display the

Table 4. Comparison of ε_z^*.

ξ	Exact	Direct HOST	Statically equivalent estimate	
			HOST	CLPT
0·2	−0·013	−0·035	−0·015	−0·024
0·4	−0·027	−0·071	−0·032	−0·048
0·5	$\begin{cases}-0{\cdot}034\\-0{\cdot}058\end{cases}$	−0·088	$\begin{cases}-0{\cdot}043\\-0{\cdot}074\end{cases}$	$\begin{cases}-0{\cdot}060\\-0{\cdot}087\end{cases}$
0·6	−0·086	−0·106	−0·094	−0·105
0·8	−0·151	−0·141	−0·149	−0·149
1·0	−0·236	−0·176	−0·228	−0·205

required discontinuity in strains at the interface and agree closely with the exact solution.

Thus we see, that by choosing statically equivalent estimates of interlaminar stresses and strains and direct estimates of in-plane stresses and strains, displacement-based models can be used to study stresses in laminates. Nevertheless from (5) it may be noted that in view of the availability of only a single constant of integration, the interlaminar stresses estimates may not, in general, satisfy plate boundary conditions at both the plate surfaces, $z = \pm h$. However, experience based on numerical studies (Krishna Murty & Vellaichamy 1988b) indicates that this violation is minimal and may not be of real practical consequence.

3. Iterative modelling

As an extension to 'displacement-based modelling', it is possible to construct a heirarchy of models, wherein the displacement field at a given stage of the iterative model is deduced from the statical equivalent strains corresponding to the previous step of iteration. Such models are referred to here as 'iterative models'. Further, the difficulty in satisfying the second plate surface boundary condition may also be removed by obtaining the basic displacement variables, $u_i, v_i, w_i \ldots$ etc. as a solution of the differential equations representing the plate surface boundary conditions instead of the "elemental equilibrium equations", whose satisfaction is not in any way essential, since local equilibrum equations are used to obtain the interlaminar stresses. Recently Valisetty & Rehfield (1985) presented a comprehensive model wherein this new hypothesis, namely that statically equivalent stresses from classical theory can be used to obtain transverse shear and normal strains, was introduced in place of the traditional Kirchhoff's hypothesis of zero transverse shear and normal strains. Krishna Murty & Vijaya Kumar (1987) and Vijaya Kumar & Krishna Murty (1988b) introduced the idea of utilizing plate surface boundary conditions, as governing equations, instead of the 'elemental equilibrium equations'.

A comparison of the displacements and stresses in a $[0/90]_s$ laminated strip subjected to sinusoidal loading of $(\sigma_z)_{z=\pm h} = \pm q_0 \sin \pi x/a$ is given in table 5. Higher accuracies attained at the second stage of the iteration are evident from this comparison.

Table 5. Comparison of displacements and stresses.

Quantity	ξ	CLPT	HOST[1]	Iterative[2]	Exact
W^*	0	1·0	1·51	1·56	1·56
	1	1·0	1·51	1·56	1·56
U^*	1	0·126	0·135	0·138	0·138
σ_{x/q_0}	0·5	$\begin{cases} 2·16 \\ 53·95 \end{cases}$	$\begin{cases} 2·00 \\ 49·22 \end{cases}$	$\begin{cases} 1·82 \\ 44·11 \end{cases}$	$\begin{cases} 1·83 \\ 43·92 \end{cases}$
	1	107·9	115·9	118·7	118·5
σ_{xz/q_0}	0·5	5·09	5·05	5·02	5·02
σ_{z/q_0}	0·5	0·323	0·320	0·318	0·318

$W^* = W/W_{CLPT}$; $U^* = U/U_{CLPT}$; [1]Krishna Murty & Vellaichamy (1988b); [2]Krishna Murty & Vijaya Kumar (1987).

4. Finite element modelling

The need for converting these models into a finite element form in order to be able to apply them to general laminates is obvious. In the displacement methods broadly two approaches may be considered for this purpose, namely (i) direct utilization of the three-dimensional finite elements, and (ii) finite elements based on classical laminated plate theory and higher order models. Isoparametric elasticity elements have been successfully used to study stresses in laminates (Whitcomb *et al* 1982; Carlsson 1983). Apart from the requirement of large degrees of freedom to model the necessary detail, it is observed (Whitcomb *et al* 1982) that the finite element solution is accurate except in one or two elements closest to the plane of discontinuity. The interface in a laminate is a plane of discontinuity in interlaminar strains. Further, isoparametric brick elements have no provision for inter-element stress continuity. Thus the suitability of such elements, for estimating interlaminar stresses which are continuous across interfaces, needs to be examined carefully. On the other hand, finite elements based on CLPT or HOST are relatively simple, and the provision to realize the necessary continuity in stresses and strains at the interfaces can be incorporated. Recently Kant (Kant 1988; Kant & Pandya 1988) has developed finite elements based on higher order theories and demonstrated their application through several examples. It will be interesting to study the performance of such elements in estimating interlaminar stresses at free edges and rivet holes to bring out the adequacy or otherwise of such elements for critical applications involving the estimation of interlaminar stresses.

5. Conclusions

In this paper displacement-based modelling of laminated composite plates is reviewed. Iterative modelling appears to be the most promising approach for the analysis of laminates as it has the provision to achieve the necessary continuities and discontinuities in stresses and strains across the laminate interfaces. The first step of the iterative model corresponds to the classical laminate plate theory with a small modification, namely statically equivalent estimates of transverse normal and shear strain replace

Kirchhoff's assumption of zero normal and shear strains. Several finite elements currently available in the literature may be easily modified to represent the first step of the iterative model. A similar approach can be implemented with finite element analysis based on higher order models also.

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