

# A hybrid approach for aeroacoustic analysis of the engine exhaust system

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## Abstract

This paper presents a new hybrid approach for prediction of noise radiation from engine exhaust systems. It couples the time domain analysis of the engine and the frequency domain analysis of the muffler, and has the advantages of both. In this approach, cylinder/cavity is analyzed in the time domain to calculate the exhaust mass flux history at the exhaust valve by means of the method of characteristics, avoiding the tedious procedure of interpolation at every mesh point and solving a number of equations simultaneously at every junction. This is done by making use of an interrelationship between progressive wave variables of the linear acoustic theory and those of the method of characteristics. In this approach, nonlinear propagation in the exhaust pipe is neglected and free expansion is assumed at the radiation end of the exhaust pipe. In the case of a muffler proper, expansion from the exhaust pipe into the first chamber is assumed to be a free expansion. Various results of this approach are compared with those of the method of characteristics and the classical acoustic theory, and various peaks and troughs in insertion loss curves are analytically validated. © 2000 Elsevier Science Ltd. All rights reserved.

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## 1. Introduction

The design and analysis of mufflers for exhaust systems has been an active area of research in recent years. For a suitable design of mufflers, the noise generating sources have to be analyzed integrally with the mufflers. Predicting radiated noise requires a model of the acoustic behavior of the intake/exhaust system and a model of the engine cycle source characteristics. This is often dealt with in either the frequency

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## Nomenclature

$A$	nondimensionalized speed of sound = $\frac{a}{a_{\text{ref}}}$
$A_0$	entropy level
$a$	sound speed
$B$	acoustic rearward wave variable
$C_p$	specific heat at constant pressure
$C_v$	specific heat at constant volume
$D$	diameter
$E$	polynomial coefficient for the exhaust area
$F$	Froude's friction factor
$f$	friction factor
$I$	polynomial coefficient for the inlet area
IL	insertion loss
$j$	$\sqrt{-1}$
$k$	wave number
$L$	reference length
$L_w$	acoustic power level
$l$	muffler element length
$M$	mean flow Mach number
MOC - 2	two characteristics approach
MOC - 3	three characteristics approach
$\dot{m}$	mass flow rate
$P$	forward Riemann variable
$p$	acoustic pressure; pressure
$Q$	rearward Riemann variable
$q$	rate of heat transfer per unit mass
$R$	radiation resistance
$R$	specific gas constant
$r$	crank radius
$S$	acoustic state vector
$S$	area of cross section of a pipe
$T$	transfer matrix
$T$	temperature; time period
$t$	time variable
$U$	mean flow velocity
$u$	velocity; acoustic particle velocity
VR	velocity ratio
$v$	acoustic mass velocity
$W$	acoustic power
$X$	nondimensionalized space variable
$x$	space coordinate
$Y$	characteristic impedance of the pipe

$Z$	nondimensionalized time variable; impedance
$z$	space coordinate
$\gamma$	ratio of specific heats
$\theta$	crank angle
$\rho$	density
$\tau$	time
$\omega$	circular frequency

domain analysis or in the time domain analysis. The frequency domain analysis of mufflers is done by means of the transfer matrix method.

The frequency domain modeling is simple and ideally suited for synthesis of a muffler configuration. However, it requires a prior knowledge of the source characteristics, the evaluation of which remains a challenge [1–8]. The time domain modeling by means of the method of characteristics is complete in itself, but is comparatively much more cumbersome. The two characteristics stationary frame method has been developed for some complex muffler elements like extended-tube resonators and perforated elements, whereas the three characteristics solution has been limited to simple uniform pipes and simple area discontinuities only [9–15]. The model has been primarily developed and used for thermodynamic performance evaluation and not for the aeroacoustic performance evaluation of the exhaust system [9].

In an attempt to overcome these disadvantages, hybrid approaches have been developed which essentially try to couple the acoustic description of the muffler piping system to the acoustic source more realistically than that given by the usual time invariant linear model. In this approach, exhaust mass flux history at the exhaust valve is calculated by the method of characteristics and it is used as an input to the frequency domain analysis of the exhaust muffler.

The most simple of these hybrid approaches is that discussed by Jones et al. [16] who studied the possibility of using a nonlinear calculation to determine the instantaneous volume velocity at the source and then applying the linear acoustic description of the piping system in order to compute the volume velocity at the open end. This simple solution represents an obvious limitation, since the first nonlinear calculation cannot take into consideration the information related to the acoustic behavior of the piping system. In this way no account is taken for the interaction between the source and the system.

Later on, Munjal [1], suggested a different hybrid approach in which the flow in the exhaust manifold and the exhaust pipe would be solved by means of the method of characteristics with the boundary condition given by the impedance of the rest of the exhaust system (muffler proper and tail pipe). This boundary condition would be imposed by assuming that initially the flow is at rest in the exhaust manifold and considering an open end boundary condition at the beginning of the muffler for the first calculation cycle.

Davies et al. [17] coupled the frequency response of an open end to the flow calculation in the duct, making use of the inverse Fourier transform of the termination

impedance and an acoustic filter placed downstream of the exhaust pipe. Later on they came up with a new hybrid approach in which pressure and velocity in the time domain are coupled with those in the frequency domain at the interface [18,19].

Payri et al. [20] came up with a new hybrid approach in which the coupling between the time domain and frequency domain is performed in such a way that the acoustic representation of the singularity is imposed as a boundary condition for a conventional time domain calculation performed with the method of characteristics. In this approach, nonlinear calculations are performed both upstream and downstream of the singularity, so that the flow in the ducts between silencer and the tail pipe is also computed by means of the method of characteristics. But in this approach, each time Fourier and inverse Fourier transforms are calculated to couple the progressive variables of both the domains.

The hybrid approaches developed recently attempt to combine the frequency domain approach with the finite wave time domain approach with a view to incorporate the advantages of both. However, they tend to be rather cumbersome because of the repeated use of the Fourier transform pair.

In this paper, time domain analysis and frequency domain analysis are coupled, so as to get rid of the disadvantages of both the methods, and yet enjoy the advantages of both. This is done by making use of an interrelationship between progressive wave variables of the linear acoustic theory and those of the method of characteristics. In this approach, nonlinear propagation in the exhaust pipe is neglected and free expansion is assumed at the radiation end of the exhaust pipe. In the case of a muffler proper, expansion from the exhaust pipe into the first chamber is assumed to be a free expansion. This is how it differs from other hybrid approaches. Results of this novel hybrid approach are compared with those of the pure aeroacoustic theory and the method of characteristics. In the hybrid approach, progressive wave variables of the linear acoustic theory and the method of characteristics are coupled, which avoids the need of the source characteristics in the frequency domain analysis and tedious meshing procedure and iterations in the time domain analysis.

## 2. Theory of the hybrid approach

Acoustic pressure and volume velocity may be written in terms of the progressive wave variables  $A$  and  $B$ , where  $A$  represents the forward wave and  $B$ , the reflected wave [1]. The corresponding time domain variables are denoted by  $P$  and  $Q$ , respectively, in this paper. We have two equations in hand to get these interrelationships. These are particle velocity and pressure equations.

In the time domain,

$$u = a_{\text{ref}} U = a_{\text{ref}} \frac{P - Q}{\gamma - 1}, \quad (1)$$

and in the frequency domain,

$$u = \frac{A - B}{\rho_0 a_0}. \quad (2)$$

It should be remembered that the variables  $A$  and  $B$  hold in the time domain as well as in the frequency domain, whereas  $P$  and  $Q$  hold in the time domain only.  $A$  and  $B$  are special cases of the time domain variables. Conversely,  $P$  and  $Q$  should be Fourier transformed variables like  $A$  and  $B$ .

In the absence of any acoustic perturbation or unsteadiness,  $A = B = 0$ , and  $P = Q = 1$ . From Eqs. (1) and (2), assuming that  $a_{\text{ref}} = a_0$  and  $B \ll p_0$ , we can interrelate these variables as follows:

$$A = \frac{\rho_0 a_0 a_{\text{ref}}}{\gamma - 1} (P - 1) = \frac{\gamma p_0}{\gamma - 1} (P - 1) \quad (3)$$

and

$$B = \frac{\rho_0 a_0 a_{\text{ref}}}{\gamma - 1} (Q - 1) = \frac{\gamma p_0}{\gamma - 1} (Q - 1). \quad (4)$$

Conversely,

$$P = 1 + \frac{\gamma - 1}{\gamma} \frac{A}{p_0} \quad (5)$$

and

$$Q = 1 + \frac{\gamma - 1}{\gamma} \frac{B}{p_0}. \quad (6)$$

To check whether these relations are consistent or not, an acoustic pressure equation may be used.

Total pressure in terms of characteristic variables  $P$  and  $Q$  can be written as (see Appendix B)

$$p_t = p_0 \left( \frac{P + Q}{2} \right)^{\frac{2\gamma}{\gamma-1}}.$$

On substituting Eqs. (5) and (6) in the above equation, yields

$$p_t = p_0 \left\{ \frac{\left( \left( 1 + \frac{\gamma - 1}{\gamma} \frac{A}{p_0} \right) + \left( 1 + \frac{\gamma - 1}{\gamma} \frac{B}{p_0} \right) \right)}{2} \right\}^{\frac{2\gamma}{\gamma-1}}$$

or,

$$p_0 + p = p_0 \left\{ 1 + \frac{\gamma - 1}{2\gamma} \frac{A + B}{p_0} \right\}^{\frac{2\gamma}{\gamma-1}}.$$

Expanding the right hand side as a series, and considering only the first order terms,

$$p_0 + p = p_0 \left\{ 1 + \frac{2\gamma}{\gamma - 1} \frac{\gamma - 1}{2\gamma} \frac{A + B}{p_0} \right\}$$

or,

$$p = A + B. \quad (7)$$

Eq. (7) is indeed the acoustic pressure equation. Hence, it is confirmed that Eqs. (3) and (4), and hence Eqs. (5) and (6), are consistent in the low amplitude limit. Somewhat similar relations have also been developed by Payri et al. [20].

### 3. Boundary conditions

Boundary condition calculations using the method of characteristics are much more cumbersome and these can be replaced by the more convenient frequency dependent boundary conditions. Thermodynamics of the cylinder/cavity is characterized as explained in Refs. [1] and [14]. Here the rate of change of pressure, temperature and density can be calculated. A cavity pipe junction boundary condition is also formulated as explained in the same references, using the two characteristics approach (MOC - 2) as outlined in Appendix B. But, in the time domain analysis, reflected variable  $Q$  can be calculated using the interpolation scheme. In the hybrid approach also, some technique should be used to update the variable  $Q$ . It is developed mainly to avoid the use of meshing and interpolation in the pipe. This is explained below.

Neglecting the effects of nonlinear propagation in the exhaust pipe and assuming free expansion at the radiation end of the exhaust pipe (Fig. 1), as the effect of radiation condition/impedance would be negligible, the fluctuating pressure would be given by,

$$A(t - \tau) + B(t) = 0 \quad (8)$$

or

$$B(t) = -A(t - \tau), \quad (9)$$

at the valve end, where  $\tau$  is the time taken by the forward progressive wave to traverse the exhaust pipe up and down. It can symbolically be written as

$$\tau = 2 \frac{l_{ex}}{a_0}, \quad (10)$$

where  $a_0$  is the reference sound speed at the hypothetical temperature obtained by isentropic expansion from the blowdown pressure to the ambient pressure.

Substituting Eqs. (3) and (4) in Eq. (9) gives

$$\frac{\gamma P_0}{\gamma - 1}(Q(t) - 1) = -\frac{\gamma P_0}{\gamma - 1}(P(t - \tau) - 1)$$

or,

$$Q(t) = 2 - P(t - \tau) \quad (11)$$

This means that  $Q$  at the present instant of time can be evaluated by using the  $P$  wave that started  $\tau$  seconds ago. This way, meshing and interpolation in the pipe can be avoided and all the attention will be limited to the valve end only.

Eq. (10) implies that sound speed remains constant at  $a_0$ . In reality, the forward wave would move with a speed of  $a_{0+u}$  and reflected wave with  $a_0 - u$ . Thus, the total up and down time  $\tau$  may be calculated with reasonable accuracy by means of Eq. (10).

During the initial phase of the first cycle of calculations, we can take

$$Q(t) = 1 \text{ for } t \leq \tau \quad (12)$$

In the case of a muffler proper, expansion from the exhaust pipe into the first chamber can be assumed to be free expansion. Thus, Eqs. (10)–(12) may still be applicable with a new value of  $l_{\text{ex}}$  and hence the time lag  $\tau$ . The same analysis can be used on the intake side also.

#### 4. Implementation of the hybrid approach

The main program for the hybrid approach is built around the cylinder. The value of  $a_{\text{ref}}$  is calculated assuming isentropic expansion from the blowdown conditions to the atmospheric pressure. The time interval is calculated by dividing the complete cycle process into 1024 equal parts. During the initial phase of the first cycle,  $Q$  is initialized to unity as shown in Eq. (12). Then  $Q$  is updated at other instants using Eq. (11). With this value of  $Q$ , the new or updated value of  $P$  can be calculated from

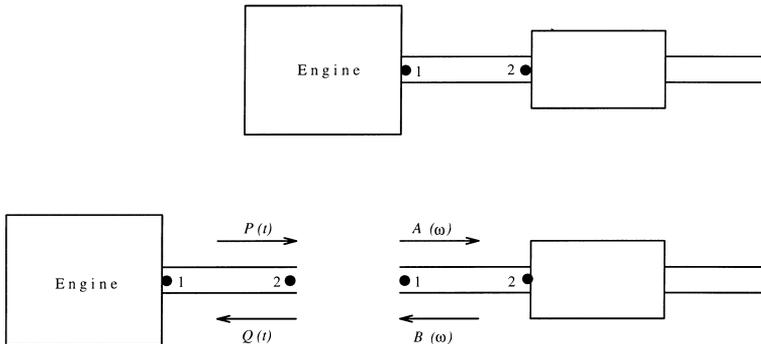


Fig. 1. Schematic representation of the hybrid approach.

the two characteristics cavity pipe junction subroutine. This is repeated for the next one or two cycles to get the steady-state flow history.

Thus, at all instants of time, the mass flux  $\dot{m}(t)$  from the valve end is calculated using values of the characteristic variables  $P$  and  $Q$  and Eq. (B26). As explained in Ref. [1], taking Fourier transform of  $\dot{m}(t)$ ,  $v_{c,n}(\omega)$  is obtained, where  $v_{c,n}$  is the convective mass velocity variable. This may be combined with the linear frequency domain model of the exhaust system as follows.

Taking overall transfer matrix of the entire muffler system, we get (see Appendix A),

$$\begin{Bmatrix} p_{c,n} \\ v_{c,n} \end{Bmatrix} = \begin{bmatrix} T_{c,11} & T_{c,12} \\ T_{c,21} & T_{c,22} \end{bmatrix} \begin{Bmatrix} p_{c,0} \\ v_{c,0} \end{Bmatrix} \quad (13)$$

or,

$$\begin{Bmatrix} p_{c,n} \\ v_{c,n} \end{Bmatrix} = \begin{bmatrix} T_{c,11} & T_{c,12} \\ T_{c,21} & T_{c,22} \end{bmatrix} \begin{bmatrix} 1 & Z_{c,0} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ v_{c,0} \end{Bmatrix}, \quad (14)$$

where

$$Z_{c,0} = Y_0 \left( \frac{K_0^2 r_0^2}{4} + j0.6k_0 r_0 + \frac{M}{2} \right).$$

In Eq. (14),  $v_{c,n}$  is known and thus  $v_{c,0}$  is calculated. Implicitly,  $v_{c,n}$  is assumed to be independent of the acoustic load (impedance) of the muffler. To that extent, this approach presumes a constant velocity or high impedance source. The convective mass velocity  $v_{c,0}$  is then used to calculate the power radiated from the tail pipe end. The procedure is repeated with and without the given muffler to calculate insertion loss of the muffler.

## 5. Computed results and discussions

The concept of the method of characteristics is to solve the given set of equations under various conditions until the solution stabilizes for a particular set of conditions. This is the biggest drawback in the system as the solutions take a number of iterations to stabilize. The severity of this problem further increases as the attached muffler configurations become more complicated. However, in the hybrid approach, it has been observed that the solutions stabilize after the first iteration itself, and very minor changes were observed on going up to as high as five iterations.

Using the computational procedure described above, an exhaust system of a single cylinder, four stroke, naturally aspirating diesel engine is simulated. Specifications of the engine are given in Appendix C. Various configurations of the exhaust muffler investigated here are shown in Fig. 2. Following Gupta's experimental observations, the blowdown pressure and temperature were taken as 0.3436 MPa and 1300 K,

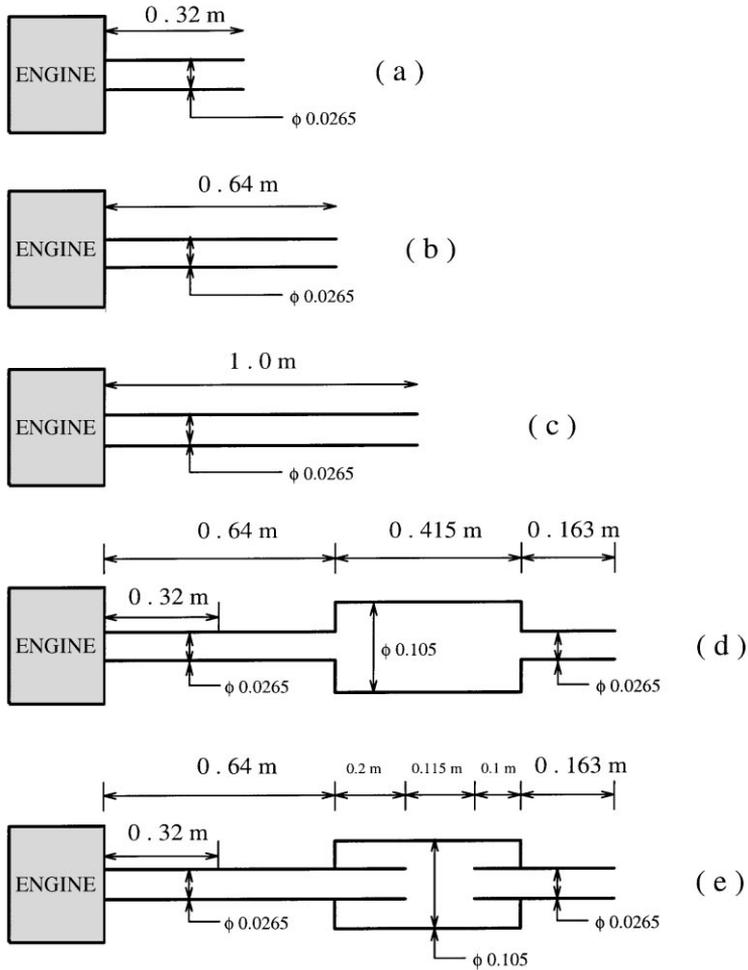


Fig. 2. The various muffler configurations investigated here.

respectively for all the configurations [5]. The blowdown conditions would not be affected by the muffler configuration downstream because the pressure ratio is more than the critical value; the wave can move downstream only. The results obtained by the hybrid method are compared with those obtained by the two characteristics (MOC -2) and the three characteristics (MOC -3) approach and also with the pure acoustic theory. In both the approaches (MOC -2 and MOC -3), the method of characteristics is used throughout the system including the exhaust pipe work.

Generally, the muffler has a small diameter pipe on either end. The one upstream pipe is called the exhaust pipe and that downstream is called the tail pipe. The middle, larger diameter portion is called the muffler proper. The basic system here is defined as the engine without any muffler proper. It consists only of exhaust pipe. In this

simulation, the basic system is taken as the exhaust pipe of length 0.32 m and diameter of 0.0265 m. This is shown in Fig. 2 (a). This is needed for calculating insertion loss of a muffler configuration where we need the sound power radiated by the engine with muffler and without muffler. Here, the term “without muffler” means “with the basic exhaust system attached to the engine”. Exhaust mass flux history, sound pressure levels and insertion loss are plotted for various configurations shown in Fig. 2.

In Figs. 3–5, exhaust mass flux histories are plotted for simple pipe lengths of 0.32, 0.64 and 1 m, respectively. In the hybrid approach, the time lag  $\tau$  is calculated with respective lengths of the pipes. In these three figures, we can observe a very good agreement between the characteristics approach and the hybrid approach. As the pipe length increases, we can see a slight variation in the three characteristics approach and the hybrid approach, whereas there is no variation in the two characteristics approach and the hybrid approach.

Exhaust mass flux histories for a simple expansion chamber and an extended-tube expansion chamber are plotted in Figs. 6 and 7. For these muffler elements, time lag  $\tau$  corresponds to the length of the exhaust pipe, i.e. 0.64 and 0.84 m respectively. It may be observed that the hybrid approach is quite satisfactory even for simple and extended-tube chambers.

With these observations, it can be said that the thermodynamic analysis of an engine with the hybrid approach is quite satisfactory when it is compared with the method of characteristics. In other words, the effect of nonlinearities is only marginal. Tedious procedure of interpolation at every mesh point and solving a number of equations simultaneously at every junction, are avoided in this approach, which saves much time and effort.

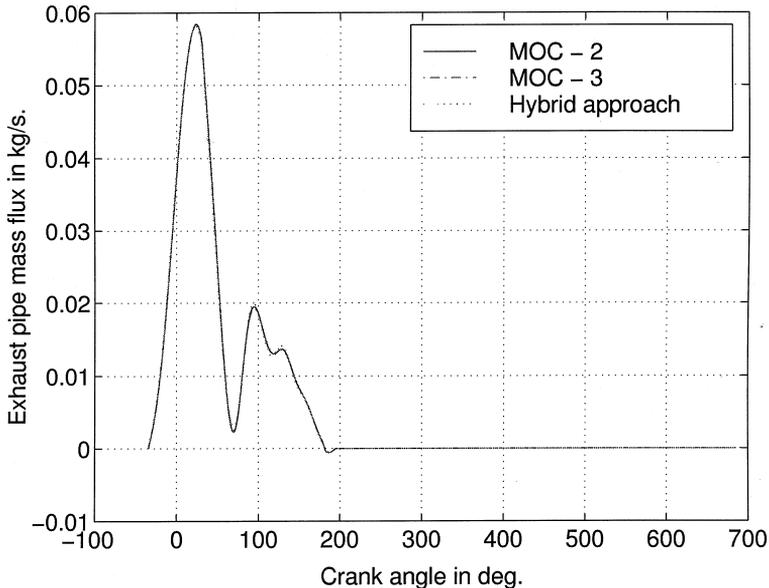


Fig. 3. Exhaust mass flux history of pipe of length 0.32 m.

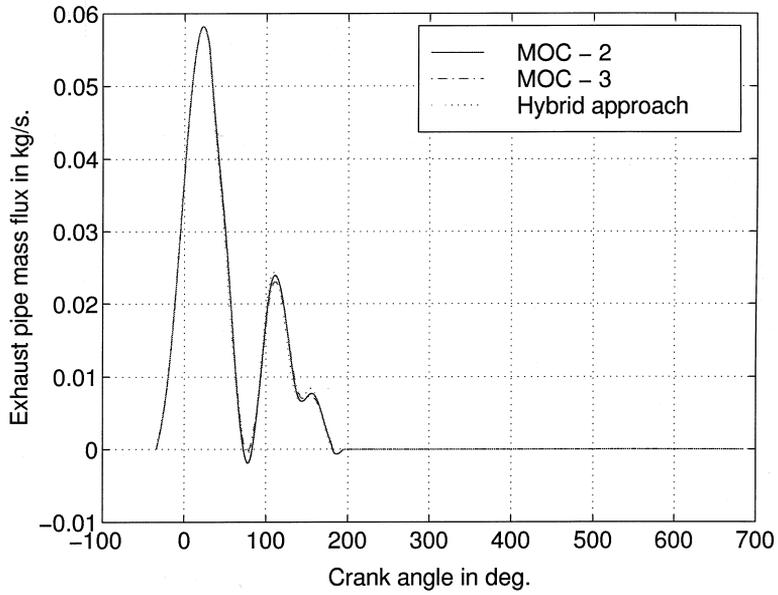


Fig. 4. Exhaust mass flux history of pipe of length 0.64 m.

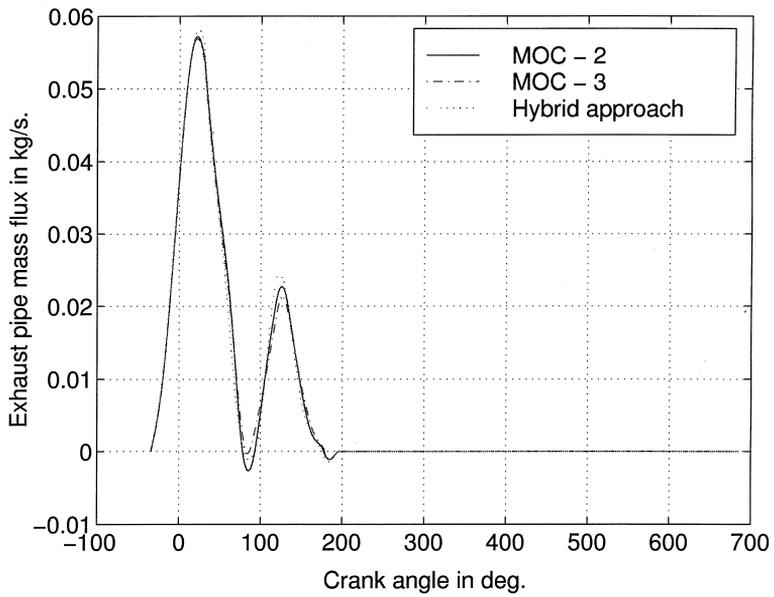


Fig. 5. Exhaust mass flux history of pipe of length 1.0 m.

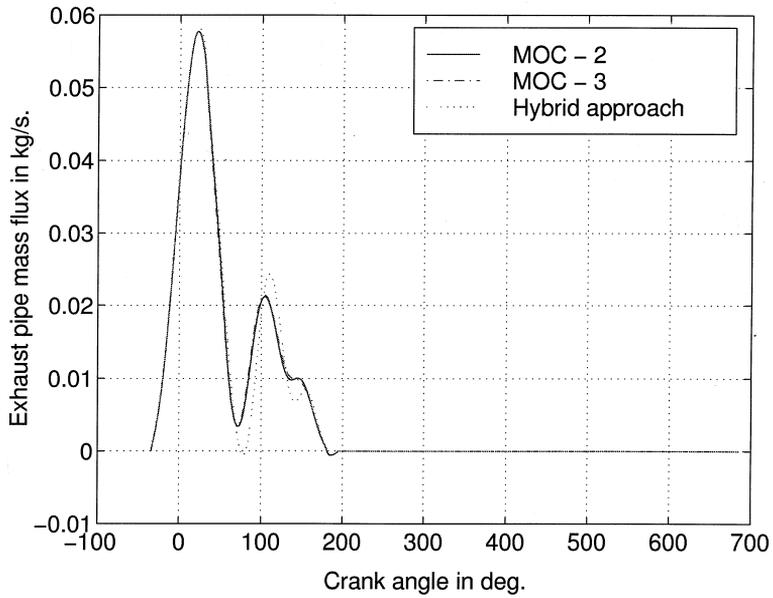


Fig. 6. Exhaust mass flux history of a simple expansion chamber muffler.

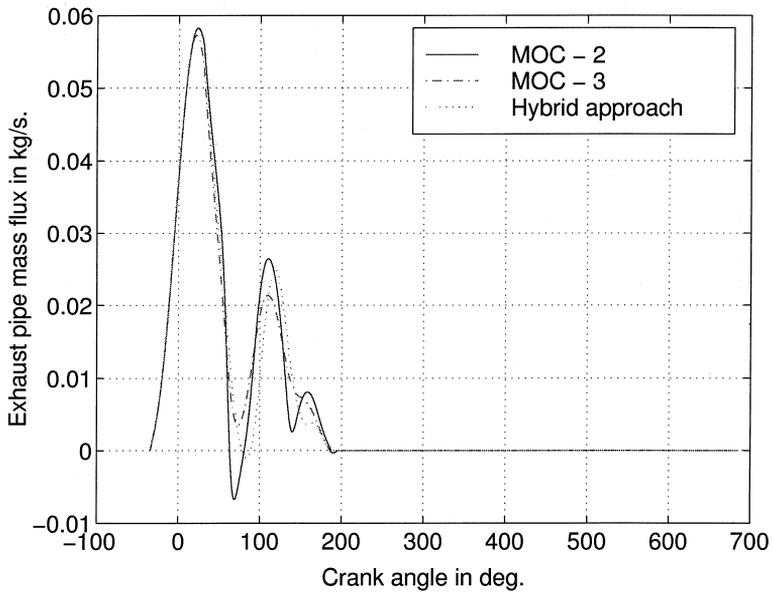


Fig. 7. Exhaust mass flux history of an extended-tube expansion chamber muffler.

Acoustic performance of the given muffler configurations is plotted in Figs. 8–14. Sound pressure levels, calculated at a distance of 1 m from the radiation end for all the configurations shown in Fig. 2, are shown in Figs. 8–12. As explained earlier, taking Fourier transform of the exhaust mass flux history, mass velocity ( $v_{c,n}(\omega)$ ), that is entering into the load, i.e. the muffler, is obtained. From this tail pipe mass velocity ( $v_{c,0}(\omega)$ ) is calculated using the transfer matrix concept. Sound pressure levels are calculated using  $v_{c,0}(\omega)$  as explained in Ref. [1]. The advantage of this approach over the classical theory of acoustics is that here sound pressure levels are also calculated as shown in Figs. 8–12.

Insertion loss values for simple chamber and extended-tube resonator mufflers are shown in Figs. 13 and 14. The results of the hybrid approach are compared with those of the pure acoustic theory, where the source is approximated as a constant velocity source (infinite source impedance). As indicated earlier, the hybrid approach, too, implies a constant velocity source. Prediction of the hybrid approach and pure acoustic theory are seen to be close to each other. It may be observed that essentially the same peaks and troughs characterize both the approaches. This can be explained as follows.

For a simple expansion chamber of length  $l_{ch}$ , insertion loss curve peaks when

$$kl_{ch} = (2n + 1)\frac{\pi}{2}$$

or when

$$f = (2n + 1)\frac{c}{4l_{ch}}, n = 0, 1, 2, \dots$$

and troughs when

$$kl_{ch} = n\pi$$

or when

$$f = n\frac{c}{2l_{ch}}, n = 1, 2, 3, \dots$$

where,

$$k = \text{Wave number} = \frac{\omega}{c}$$

$c$  = Sound speed (here, it is equal to  $a_{ref}$ )

$\omega$  = Circular frequency

$f$  = Frequency ( $mf_0$ )

$f_0$  = Firing frequency (12.5 Hz).

The same can be applied to the extended-tube resonators and tail pipe replacing the length  $l_{ch}$  by the corresponding lengths, i.e. extended-tube length  $l_{ex}$  and tail pipe length  $l_{tp}$ .

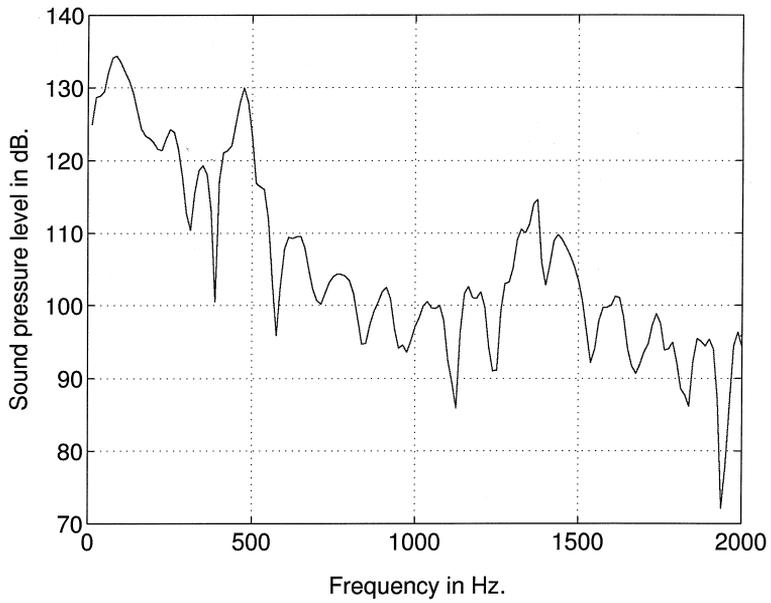


Fig. 8. Sound pressure level at a distance of 1 m from the radiation end of a simple pipe of length 0.32 m.

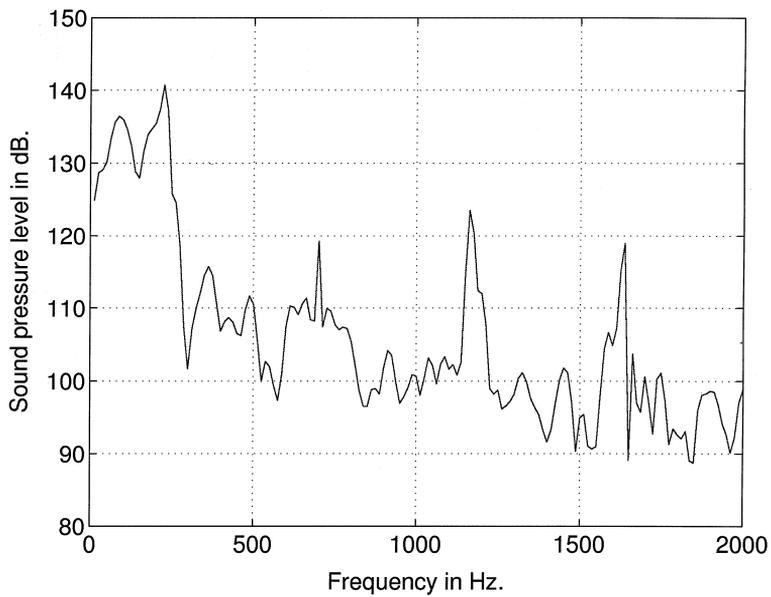


Fig. 9. Sound pressure level at a distance of 1 m from the radiation end of a simple pipe of length 0.64 m.

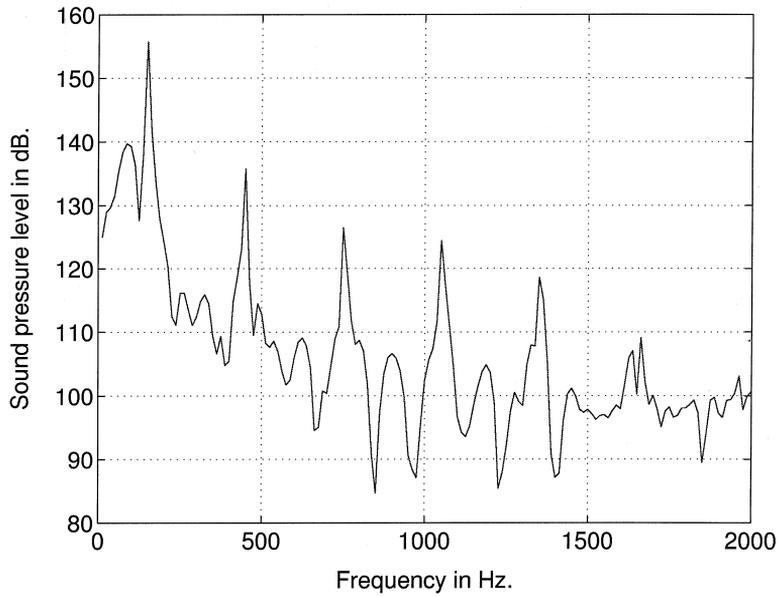


Fig. 10. Sound pressure level at a distance of 1 m from the radiation end of a simple pipe of length 1.0 m.

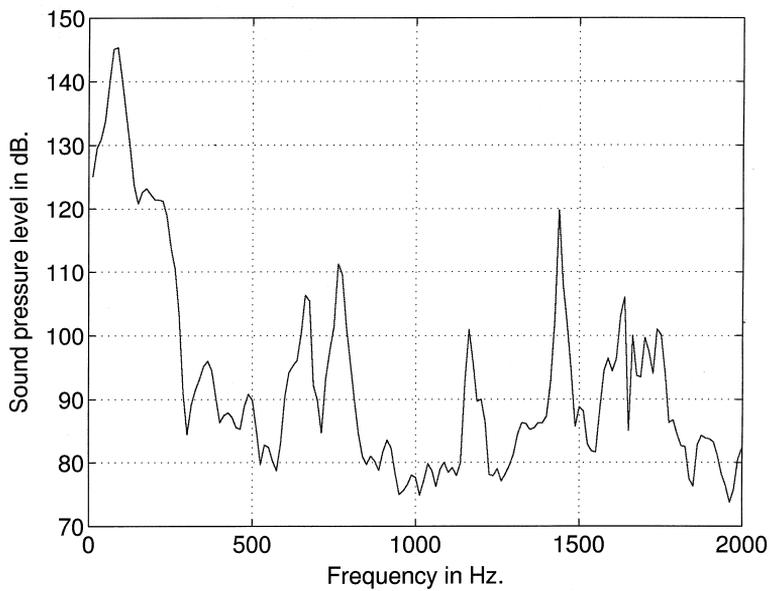


Fig. 11. Sound pressure level at a distance of 1 m from the radiation end of a simple expansion chamber muffler.

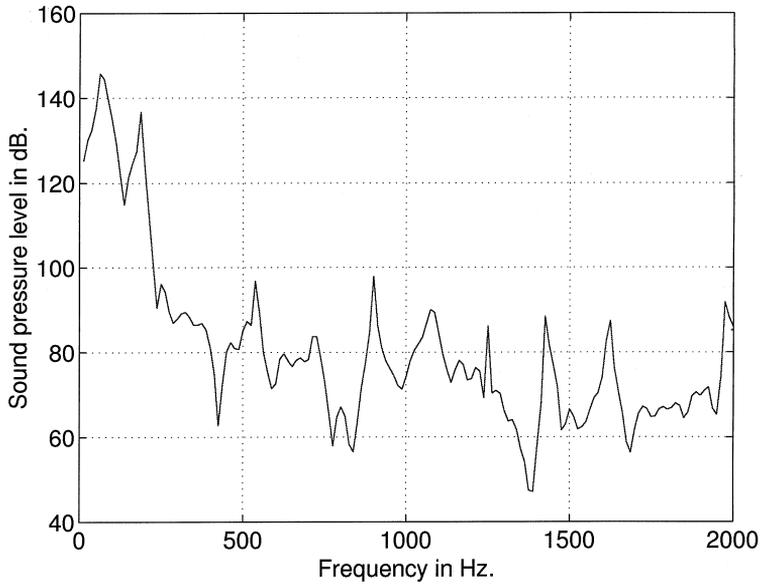


Fig. 12. Sound pressure level at a distance of 1 m from the radiation end of an extended-tube expansion chamber muffler.

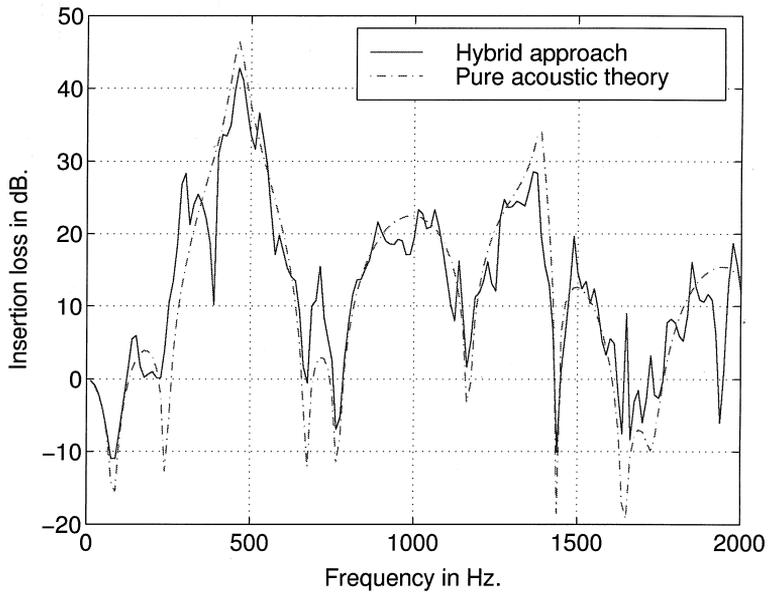


Fig. 13. Insertion loss of a simple expansion chamber muffler.

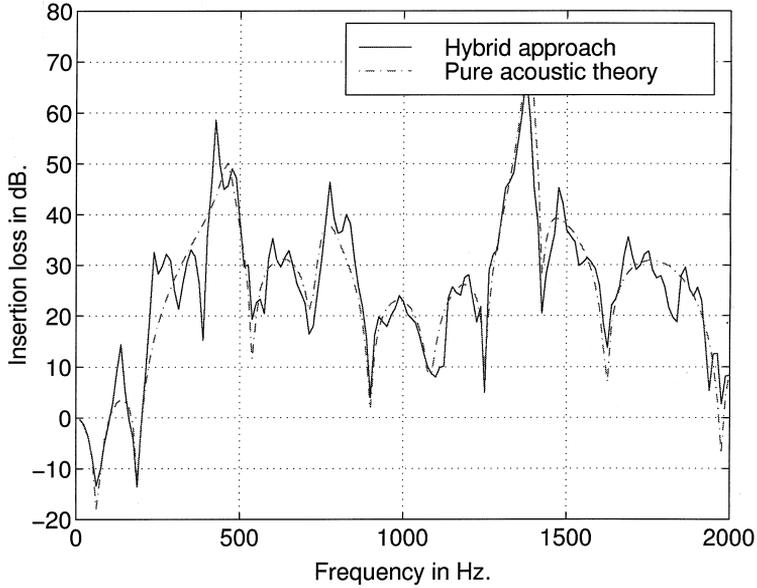


Fig. 14. Insertion loss of an extended-tube expansion chamber muffler.

In the present investigation, simple expansion chamber length,  $l_{ch}$  is taken as 0.415 m and the corresponding tail pipe length,  $l_{tp}$  is 0.163 m. In the extended-tube expansion muffler,  $l_{ex}$  values are taken as 0.2 and 0.1 m, and the tail pipe length,  $l_{tp}$  is 0.263 m. Sound speed  $c$  is 608 m/s.

In Fig. 13, we can observe the peaks and troughs of the insertion loss curve for a simple expansion chamber obtained by the classical acoustic theory and the hybrid approach. According to the above equations, peaks should occur at frequencies 366, 1098, 1830 and 932 Hz, and troughs at 732, 1464 and 1864 Hz. In this figure, we can observe that the peaks and troughs roughly correspond to these frequencies.

Similarly, in Fig. 14, insertion loss curve for the extended-tube expansion chamber is shown. Theoretically peaks should occur at frequencies 760, 578 and 1734 Hz, and troughs at 1155 and 1520 Hz. In this figure also, we can observe that the peaks and troughs are in the neighborhood of these frequencies.

Thus, results of the hybrid approach tally quite well with those of the time domain approach (Figs. 3–7) and the frequency domain approach (Figs. 13 and 14).

## 6. Concluding remarks

The hybrid approach presented above is a new approach making use of the time domain modeling of the cylinder and cavity pipe junction, and the linear frequency domain analysis of the muffler. The hybrid approach is observed to be a very good coupler between the time domain and frequency domain, combining advantages of both. The hybrid approach presented here has the advantage of simplicity over those

available in the literature. However, it presumes the presence of an expansion chamber or cavity downstream of the exhaust pipe, and models it as free expansion.

Mass flux histories of the various configurations obtained by means of the hybrid approach are compared with those obtained by means of the two characteristics and the three characteristics approach. It is observed that the discrepancies among the results obtained by these three approaches are not substantial.

Insertion loss plots for a simple expansion chamber and an extended-tube expansion chamber predicted with the hybrid approach and classical acoustic theory reveal that the peaks and troughs obtained by the hybrid approach generally coincide with those obtained by the acoustic theory, thereby providing a validation of the hybrid approach.

## Appendix A: Linear acoustic theory of mufflers

This appendix is adopted from Chapters 2, 3, 5 and 7 of Ref. [1]. In the classical theory of acoustic filters, where the medium is assumed to be stationary, the state of acoustic waves is characterized by two state variables, namely, acoustic pressure  $p$  and the acoustic mass velocity  $v$ . These can be represented as

$$\begin{aligned} p &= Ae^{-jkz} + Be^{+jkz} \\ v &= \frac{1}{Y}(Ae^{-jkz} - Be^{+jkz}), \end{aligned} \quad (\text{A1})$$

where  $A$  and  $B$  are amplitudes of acoustic pressure associated with the forward progressive wave and reflected progressive wave, respectively. However, in the presence of mean flow, the convective effect of the mean flow can be taken into account if one replaces the state variables  $p$  and  $v$  by the corresponding convective state variables, namely, the aeroacoustic pressure  $p_c$  and the aeroacoustic mass velocity  $v_c$ . Similar to  $p$  and  $v$  which are perturbations on static pressure and zero mean mass velocity,  $p_c$  and  $v_c$ , denote perturbations over the total pressure and the total mass velocity in the system. Thus, these aeroacoustic state variables are linearly related to their acoustic counterparts as

$$\begin{Bmatrix} p_c \\ v_c \end{Bmatrix} = \begin{bmatrix} 1 & MY \\ \frac{M}{Y} & 1 \end{bmatrix} \begin{Bmatrix} p \\ v \end{Bmatrix} \quad (\text{A2})$$

where,

$M$  is the mean flow Mach number,

$Y$  is the characteristic impedance of the tube, defined by  $Y = \frac{a}{S}$ ,

$a$  is the sound speed,

$S$  is the area of cross section of the duct, and

$k$  is the wave number, defined by  $\frac{\omega}{a}$ .

### Muffler representation

In the linear acoustic filter theory, a muffler element is represented in terms of its fourpole parameters. These parameters relate the state variables upstream of the element to those downstream of the element as

$$\begin{Bmatrix} p_u^i \\ v_u^i \end{Bmatrix} = \begin{bmatrix} T_{11}^i & T_{12}^i \\ T_{21}^i & T_{22}^i \end{bmatrix} \begin{Bmatrix} p_d^i \\ v_d^i \end{Bmatrix} \quad (\text{A3})$$

where subscripts  $u$  and  $d$  indicate upstream and downstream variables, respectively and superscript  $i$  indicates the  $i$ th element of the muffler. Similar relations exist for the aeroacoustic state variables:

$$\begin{Bmatrix} p_{c,u}^i \\ v_{c,u}^i \end{Bmatrix} = \begin{bmatrix} T_{c,11}^i & T_{c,12}^i \\ T_{c,21}^i & T_{c,22}^i \end{bmatrix} \begin{Bmatrix} p_{c,d}^i \\ v_{c,d}^i \end{Bmatrix} \quad (\text{A4})$$

where subscript  $c$  denotes convective or aeroacoustic variables. These fourpole parameters constitute the so-called transfer matrix of the element.

Denoting the vector of the acoustic variables by  $\{\mathbf{S}\}$ , the vector of aeroacoustic variables by  $\{\mathbf{S}\}_c$  and the transfer matrix by  $[\mathbf{T}]$  and  $[\mathbf{T}]_c$ , respectively, Eqs. (A3) and (A.4) can be rewritten as

$$\{\mathbf{S}\}_u^i = [\mathbf{T}]^i \{\mathbf{S}\}_d^i \quad (\text{A5})$$

and

$$\{\mathbf{S}\}_{c,u}^i = [\mathbf{T}]_c^i \{\mathbf{S}\}_{c,d}^i \quad (\text{A6})$$

respectively, where the subscripts and superscripts have the same connotations.

In a typical muffler, many elements are combined in a cascade. One can relate the state variables upstream of the muffler with those at the downstream end of the muffler. in other words,  $\{\mathbf{S}\}^n$  and  $\{\mathbf{S}\}^0$ , or  $\{\mathbf{S}\}_c^n$  and  $\{\mathbf{S}\}_c^0$  can be related. This can be symbolically expressed as

$$\{\mathbf{S}\}^n = [\mathbf{T}]^n [\mathbf{T}]^{n-1} \dots [\mathbf{T}]^r \dots [\mathbf{T}]^2 [\mathbf{T}]^1 \{\mathbf{S}\}^0 \quad (\text{A7})$$

and

$$\{\mathbf{S}\}_c^n = [\mathbf{T}]_c^n [\mathbf{T}]_c^{n-1} \dots [\mathbf{T}]_c^r \dots [\mathbf{T}]_c^2 [\mathbf{T}]_c^1 \{\mathbf{S}\}_c^0 \quad (\text{A8})$$

Thus, the overall transfer matrix of an  $n$ -element complex muffler which relates  $\{\mathbf{S}\}^n$  and  $\{\mathbf{S}\}^0$  and  $\{\mathbf{S}\}_c^n$  and  $\{\mathbf{S}\}_c^0$  can readily be obtained from the transfer matrices of the constituent elements. This can be done by simply multiplying the individual transfer matrices in the same order in which the elements appear in the muffler. Symbolically, this can be written as

$$[\mathbf{T}] = [\mathbf{T}]^n [\mathbf{T}]^{n-1} \dots [\mathbf{T}]^r \dots [\mathbf{T}]^2 [\mathbf{T}]^1, \quad (\text{A9})$$

and

$$[\mathbf{T}]_c = [\mathbf{T}]_c^n [\mathbf{T}]_c^{n-1} \dots [\mathbf{T}]_c^r \dots [\mathbf{T}]_c^2 [\mathbf{T}]_c^1 \quad (\text{A10})$$

where  $[\mathbf{T}]$  and  $[\mathbf{T}]_c$  denote the overall transfer matrix of the  $n$ -element muffler for the two sets of state variables. The relationship between  $\{\mathbf{S}\}^n$  and  $\{\mathbf{S}\}^0$ , and  $\{\mathbf{S}\}_c^n$  and  $\{\mathbf{S}\}_c^0$ , can now be written as

$$\{\mathbf{S}\}^n = [\mathbf{T}] \{\mathbf{S}\}^0 \quad (\text{A11})$$

and

$$\{\mathbf{S}\}_c^n = [\mathbf{T}]_c \{\mathbf{S}\}_c^0. \quad (\text{A12})$$

Transfer matrices for most muffler elements are available in the literature [1–8].

### *Insertion loss (IL)*

Insertion loss is defined as the difference between the acoustic power radiated without any filter and that with the filter. Symbolically,

$$\text{IL} = L_{W1} - L_{W2} = 10 \log \left( \frac{W_1}{W_2} \right) \text{dB} \quad (\text{A13})$$

where subscripts 1 and 2 denote systems without and with mufflers, respectively. The expression for insertion loss in terms of impedances and transfer matrix parameters is given by [1]

$$\text{IL} = 20 \log \left[ \left( \frac{\rho_{0,2}}{\rho_{0,1}} \right)^{\frac{1}{2}} \left( \frac{R_{c,0,1}}{R_{c,0,2}} \right)^{\frac{1}{2}} \left| \frac{Z_{c,s}}{Z_{c,0,1} + Z_{c,s}} \right| |VR_c| \right] \quad (\text{A14})$$

where the velocity ratio  $VR_c$  is equal to the second row second column element of the overall transfer matrix and the other symbols  $\rho_0$ ,  $Z_{c,0}$ ,  $R_{c,0}$  and  $Z_{c,s}$  denote the mean density, radiation impedance, radiation resistance and source impedance, respectively. The meaning of the subscripts 1 and 2 remains the same as stated above in Eq. (A13).

## **Appendix B: The method of characteristics**

The basic equations of mass continuity, momentum, energy balance and state can be manipulated to obtain the following canonical equations: [1,9]

$$\frac{dp}{dt} \pm pa \frac{du}{dt} - (\gamma - 1)\rho \left( q + u \frac{4fu^2}{D} \frac{u}{2|u|} \right) \pm \frac{4f\rho au^2}{D} \frac{u}{2|u|} = 0 \quad (\text{B1})$$

along the lines

$$\frac{dx}{dt} = u \pm a \quad (\text{B2})$$

and

$$\frac{dp}{dt} - a^2 \frac{d\rho}{dt} - (\gamma - 1)\rho \left( q + u \frac{4fu^2}{D} \frac{u}{2|u|} \right) = 0 \quad (\text{B3})$$

along the line

$$\frac{dx}{dt} = u. \quad (\text{B4})$$

All the symbols in these equations have been defined in the Nomenclature.

Following the convention, a hypothetical speed of sound  $a_0$ , such that  $a_0$  corresponds to a state that would be reached if the gas were to expand isentropically from instantaneous local pressure  $p$  to a reference pressure  $p_0$ , is introduced. Symbolically,  $a_0$  is defined by the relation

$$\frac{a}{a_0} \equiv \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{2\gamma}}. \quad (\text{B5})$$

This definition along with the Eqs. (B1–B4) and considerable algebra yields [1,9]

$$\frac{d}{dt} \left( a \pm \frac{\gamma-1}{2} u \right) = \frac{a}{a_0} \frac{da_0}{dt} + \frac{(\gamma-1)^2}{2a} \left( q + u \frac{4fu^2}{D} \frac{u}{2|u|} \right) \mp \frac{\gamma-1}{2} \frac{u^2}{2|u|} \quad (\text{B6})$$

along the lines

$$\frac{dx}{dt} = u \pm a \quad (\text{B7})$$

$$\frac{da_0}{dt} = \frac{(\gamma-1)a_0}{2a^2} \left( q + u \frac{4fu^2}{D} \frac{u}{2|u|} \right) \quad (\text{B8})$$

along the line

$$\frac{dx}{dt} = u. \quad (\text{B9})$$

Hence  $a + \frac{\gamma-1}{2}u$ ,  $a - \frac{\gamma-1}{2}u$  and  $a_0$  are the three variables along their respective paths  $\frac{dx}{dt} = u + a$ ,  $\frac{dx}{dt} = u - a$  and  $\frac{dx}{dt} = u$ .

In the two characteristics approach (without entropy variable), friction and heat transfer are neglected. The flow is assumed to be homentropic. Thus,

$$q = 0 \text{ and } f = 0 \quad (\text{B10})$$

and an arbitrary reference speed of sound  $a_{\text{ref}}$  is chosen, and everywhere

$$a_0 = a_{\text{ref}}, \quad (\text{B11})$$

or

$$\text{entropy variable } (A_0) = \frac{a_0}{a_{\text{ref}}} = 1. \quad (\text{B12})$$

At this point and nondimensional parameters  $A = a/a_{\text{ref}}$  and  $U = u/a_{\text{ref}}$  are introduced; also a reference length  $L$  is chosen and parameters  $X = x/L$  and  $Z = a_{\text{ref}}t/L$  are introduced.

Defining two new variables by

$$P \equiv A + \frac{\gamma-1}{2}U, \quad (\text{B13})$$

and

$$Q \equiv A - \frac{\gamma-1}{2}U, \quad (\text{B14})$$

and substituting them in Eqs. (B6) and (B7) along with Eqs. (B10–B12) gives

$$dP = 0 \text{ or } P = \text{constant} \quad (\text{B15})$$

along the line

$$\frac{dX}{dZ} = U + A, \quad (\text{B16})$$

and

$$dQ = 0 \text{ or } Q = \text{constant} \quad (\text{B17})$$

along the line

$$\frac{dX}{dZ} = U - A. \quad (\text{B18})$$

Thus,  $P$  and  $Q$  are the final two variables which move along the paths defined above. These directions are called the characteristic directions or simply the characteristics of the respective variables, and the variables are called the characteristic variables. All the other variables like Mach number  $M$ , pressure  $p$ , density  $\rho$ , temperature  $T$ , mass flow rate  $\dot{m}$ , etc. can be written in terms of the characteristic wave variables  $P$  and  $Q$ , and the reference values  $p_0$ ,  $a_{\text{ref}}$  and  $L$ , as follows [1]:

$$A = \frac{P + Q}{2}, \quad a = a_{\text{ref}} A = a_{\text{ref}} \frac{P + Q}{2}; \quad (\text{B19})$$

$$U = \frac{P - Q}{\gamma - 1}, \quad u = a_{\text{ref}} U = a_{\text{ref}} \frac{P - Q}{\gamma - 1}; \quad (\text{B20})$$

$$M = \frac{u}{a} = \frac{U}{A} = \frac{2}{\gamma - 1} \frac{P - Q}{P + Q}; \quad (\text{B21})$$

$$p = p_0 \left( \frac{a}{a_0} \right)^{\frac{2\gamma}{\gamma-1}} = p_0 \left( \frac{a}{a_{\text{ref}}} \right)^{\frac{2\gamma}{\gamma-1}} = p_0 \left( \frac{P + Q}{2} \right)^{\frac{2\gamma}{\gamma-1}}; \quad (\text{B22})$$

$$\rho = \frac{\gamma p}{a^2} = \frac{\gamma p}{a_{\text{ref}}^2 A^2} = \frac{\gamma p_0}{a_{\text{ref}}^2} \left( \frac{P + Q}{2} \right)^{\frac{2}{\gamma-1}}; \quad (\text{B23})$$

$$T = \frac{a^2}{\gamma R} = \frac{a_{\text{ref}}^2}{\gamma R} A^2 = \frac{a_{\text{ref}}^2}{\gamma R} \left( \frac{P + Q}{2} \right)^2; \quad (\text{B24})$$

$$T_s = T \left( 1 + \frac{\gamma - 1}{2} M^2 \right) = \frac{a_{\text{ref}}^2}{\gamma R} \left( \frac{P + Q}{2} \right)^2 \left\{ 1 + \frac{2}{\gamma - 1} \left( \frac{P - Q}{P + Q} \right)^2 \right\}; \quad (\text{B25})$$

$$\dot{m} = \rho S u = S \frac{\gamma p_0}{a_{\text{ref}}^2} \left( \frac{P + Q}{2} \right)^{\frac{2}{\gamma-1}} \frac{P - Q}{\gamma - 1}; \quad (\text{B26})$$

where  $T_s$  is stagnation temperature and  $\dot{m}$  is mass flux through a uniform tube of area  $S$ .

Eqs. (B13) to (B26) are applicable for the two-variable method of characteristics (MOC-2). For the corresponding three-variable version (MOC-3), the reader may look up Refs. [1,9,14], among others.

### Appendix C: Engine specifications

The engine used in this model is the same as used by Gupta [15]. Specifications of the engine are given below.

Kirloskar type AV1, single cylinder, water-cooled, vertical, four-stroke, open combustion chamber, naturally aspirating diesel engine.

Rated brake power	: 5 hp
Rated speed	: 1500 RPM
Compression ratio	: 16:1
Bore	: 0.08 m
Stroke	: 0.11 m
Connecting rod length	: 0.23 m

Valve timings

Exhaust valve opening (EVO)	: 34° before BDC
Inlet valve opening (IVO)	: 175° after BDC
Exhaust valve closing (EVC)	: 195° after BDC

where, BDC denotes the Bottom Dead Center.

Using the port area vs. crank angle data, a least square fit was obtained for the curve [15]. These polynomial relations were used in the simulation program to compute the inlet area and exhaust area in the cavity pipe junction.

The polynomial for the exhaust port is given by

$$\text{Exhaust area} = \sum_{i=1}^{i=7} E_i \Delta\theta^{i-1} \quad (\text{C1})$$

where  $\Delta\theta$  is given by

$$\Delta\theta = \theta - \theta_{\text{EVO}} \quad (\text{C2})$$

or by

$$\Delta\theta = \theta_{\text{EVC}} - \theta \quad (\text{C3})$$

whichever is less. The corresponding polynomial for the inlet port is given by

$$\text{Inlet area} = \sum_{i=1}^{i=8} I_i \Delta\theta^{i-1} \quad (\text{C4})$$

where  $\Delta\theta$  is given by

$$\Delta\theta = \theta - \theta_{\text{IVO}} \quad (\text{C5})$$

In both the expressions  $\theta$  is the crank angle in degrees measured from the bottom dead center (BDC) of the engine. The values of the  $\theta_{\text{EVO}}$ ,  $\theta_{\text{EVC}}$  and  $\theta_{\text{IVO}}$  are the crank angles corresponding to exhaust valve opening, exhaust valve closing and the inlet valve opening, respectively.

The values of the polynomial coefficients for the exhaust area ( $E_i$ ) and inlet area ( $I_i$ ) for the particular engine are [15]

$$\begin{array}{ll}
 E_1 = & 0.53680 \times 10^{-6} & I_1 = & - & 0.51650 \times 10^{-6} \\
 E_2 = & 2.16910 \times 10^{-6} & I_2 = & & 0.96670 \times 10^{-6} \\
 E_3 = & 0.01560 \times 10^{-6} & I_3 = & & 0.03676 \times 10^{-6} \\
 E_4 = & 0.00594 \times 10^{-6} & I_4 = & & 0.00404 \times 10^{-6} \\
 E_5 = & - & 0.10750 \times 10^{-9} & I_5 = & - & 0.23910 \times 10^{-10} \\
 E_6 = & & 0.61950 \times 10^{-12} & I_6 = & - & 0.79060 \times 10^{-12} \\
 E_7 = & - & 0.10650 \times 10^{-14} & I_7 = & & 0.92810 \times 10^{-14} \\
 & & & I_8 = & - & 0.27960 \times 10^{-16}
 \end{array}$$

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